AN INTRODUCTION

TO THE

MATHEMATICAL

THEORY OF ATTRACTION.

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BY

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PREFACE.

The Theory of Attraction owes its origin to Newton. At first its chief value lay in its connexion with Physical Astronomy. The problems of the Figure of the Earth and Precession gave rise to the splendid analysis of Laplace.

Towards the close of the last century the science of Electricity began to be developed, and imparted a new and a wider interest to the Theory of Attraction. Fresh problems were offered for solution, and new modes of treatment were devised by Physicists, among whom Gauss, Green, and Thomson have been pre-eminent. The rapid progress of the sciences of Electricity and Magnetism has been continually increasing the direct value of the Theory of Attraction, but its indirect value has become perhaps still greater.

For some time past Physical speculation has seemed to point to the Theory of Fluid Motion as the root science of nature. In recent times the investigations of Stokes, Thomson, and Helmholtz have shown that the leading problems in the Theory of Fluid Motion are mathematically the same as problems in the Theory of Attraction. This mathematical similarity is to be found also, to a considerable extent, in the Theory of Stress and Strain in Elastic Solids. The Theory of Attraction is thus the portal to most of the higher departments of Mathematical Physics, and this is so even if it should be shown that direct action at a distance does not exist in nature.

It is then a matter of much importance that the acquisition of a competent knowledge of the Theory of Attraction should be made as easy as possible. To assist in doing this is the object of this book. It is a book for Students, not for Professors. To enable the Student to economize time and labour has been my chief aim. I have borrowed from every source with which I am acquainted, but the books from which I have drawn most material are Maxwell's "Treatise on Electricity and Magnetism," Thomson and Tait's "Natural Philosophy," and Routh's "Analytical Statics."

Many mathematical investigations for which I might have referred to Treatises on Pure Mathematics, I have introduced for the purpose of rendering the progress of the student more easy. Questions as to the original discoverer of an important theorem I have not attempted to discuss. According to most French writers every discovery of any value has been made by a Frenchman, and according to some English by an Englishman. Lord Kelvin I designate as Thomson. To call him Lord Kelvin seems as absurd as it would be to speak of Bacon as Lord Verulam.

If health and opportunity be mine, I hope, at some future time, to make this book more complete by the addition of chapters dealing with Spherical Harmonics, Conjugate Functions, and the Theory of Magnetism for bodies having finite dimensions.

I have to return my best thanks to Dr. Williamson and Mr. Frederick Purser, for their kindness in reading proof-sheets, and in giving me many valuable suggestions and investigations.

FRANCIS A. TARLETON.

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The continuity of the components X, Y, Z, of the force due to a volume distribution of mass M is seen most easily in the following manner:—

Let P_1 and P_2 be two consecutive points in space; take the axis of x in the direction P_1P_2, and let X_1 and X_2 be the components of force, at the points P_1 and P_2, due to the system of mass M.

It is plain that X_2 is equal to the component of force at P_1 due to M displaced through the distance P_2P_1. Hence, $X_2 - X_1 = X'$, where X' is the component of force at P_1 due to a system of mass M' composed of the two super-imposed systems M displaced through P_2P_1 and $-M$. Accordingly M' is made up of a volume distribution whose density at any point is $\frac{d\rho}{dx}dx$, and two surface distributions whose densities are $\rho d\nu$ and $-\rho d\nu$, where $d\nu$ is the normal displacement of a point on the boundary of M due to the translational displacement dx. Since the density of M' is everywhere infinitely small, so also is X', and therefore X is continuous.

A like mode of procedure may be employed in other cases similar to the above.

THE

MATHEMATICAL THEORY OF ATTRACTION.

CHAPTER I.

INTRODUCTORY.

1. **Universal Gravitation.**—From the laws of motion

ERRATUM.

Pages 130, 149, 150, 165, 166, *for* " Mac Claurin " *read* " Maclaurin ".

2. **Electric Forces.**—With respect to electric phenomena, bodies are usually divided into Conductors and Non-conductors. A conducting body can, in various ways, be brought into a certain state in which it is said to be charged with electricity. When two bodies are so charged, a mutual force acts between them; and if the bodies be so small compared with the intervening distance that they may be regarded as points, this force is proportional to the product of the charges, and inversely proportional to the square of the intervening distance.

As regards their mutual action, charges of electricity behave therefore to a certain extent like masses of gravitating matter, but there is an important difference which is immediately manifested. In the case of gravitating matter, the mutual force between two particles is always attractive; but charges of electricity, on the other hand, sometimes attract and sometimes repel one another.

The observed phenomena can be explained by supposing that an electric charge, or in common language electricity, is one or other of two different kinds; that two charges or quantities of electricity of the same kind repel one another; and that two charges of opposite kinds attract one another, the force in each case being proportional to the product of the charges or quantities of electricity.

To express mathematically what has been said above, we must regard one kind of electricity as positive and the other kind as negative. A quantity of electricity which occupies a space so small that it may be regarded as a point may be called an electric particle; and if a force which is repulsive be considered as positive, we may enunciate the fundamental law of electric forces by saying, that *two particles of electricity act on each other with a force directly proportional to the product of their masses, and inversely proportional to the square of the distance between them.* The word 'mass' is here used only in reference to *the amount of force* which the corresponding quantity of electricity is capable of producing, and is not meant to imply inertia or exclusive occupation of space.

Looked at from a mathematical point of view, electric forces are of a more general character than those resulting from gravitation, and the study of the former includes that of the latter.

3. **Distribution of Electricity.**—As the electric forces due to the elements of an electrified body have each a definite magnitude, and pass through the elements to which they correspond, it may be said that there is a certain quantity of electricity accumulated in each space-element of the body, and we may consider the *distribution of electricity in an electrified conductor.*

4. **Magnetic Forces.**—The elementary forces due to the presence of a magnetized body are in some respects

similar to those resulting from electricity, as they may be either attractive or repulsive, but in the case of magnetism, *forces of the two kinds are always manifested simultaneously.* The observed phenomena of magnetism can be explained by supposing that equal quantities of opposite kinds of magnetism are present in every element of the magnetized body, and are located at the extremities of an infinitely short straight line whose direction is that of the magnetization of the element.

5. **Problems to be considered.**—The main problems in the Theory of Attraction are, to find the resultant force between two bodies, and to determine the distributions of Electricity and Magnetism which take place under given conditions. These problems are however, in general, so difficult that much preliminary knowledge of theorems which, in certain cases, lead indirectly to their solution, is requisite.

6. **Physical Mode of Action of Forces.**—The researches of Faraday and others have shown that the electric forces acting between two charged conductors are dependent on the intervening medium, and are not direct action at a distance. A similar proposition in the case of the forces due to gravitation seems, up to the present, to be only a hypothesis.

The mechanism by which the final action is brought about need not be considered so far as the mathematical theory of Attraction is concerned. That theory rests only on the hypothesis that there are certain forces having certain definite magnitudes and directions.

CHAPTER II.

RESULTANT FORCE.

7. **Force at a Point.**—If f denote the mutual force between unit masses when concentrated at points at the unit distance from each other, $\frac{fmm'}{\mathrm{r}^2}$ expresses the force between two particles whose masses are m and m', and whose distance apart is r. The *resultant force at a point P* due to any system of acting masses may be defined as the resultant of the forces which would be exerted by these masses on the *unit mass* if concentrated at P. Let x, y, z be the coordinates of P; ξ, η, ζ those of a point Q at which the acting mass is m, and r the distance of P from Q, then the force at P due to m is $\frac{fm}{\mathrm{r}^2}$ or $\frac{-fm}{\mathrm{r}^2}$, according as the forces under consideration are electric or gravitational. In order to find the total force at P we must resolve each elementary force into its components parallel to the axes, and find the sum of these for each axis. If X, Y, Z be the components of the resultant at P, we have, then,

$$\left.\begin{aligned} X &= \pm f\sum m\frac{x-\xi}{\mathrm{r}^3} \\ Y &= \pm f\sum m\frac{y-\eta}{\mathrm{r}^3} \\ Z &= \pm f\sum m\frac{z-\zeta}{\mathrm{r}^3} \end{aligned}\right\}. \tag{1}$$

These expressions may be simplified by taking as the unit of force the repulsion or attraction between unit masses at the unit distance apart, in which case f becomes unity. If we select any other unit of force we must introduce the corresponding value of f.

Not only for the reason mentioned in Art. 2, but also

because it is usually more convenient to count distances from the acting mass rather than towards it, *the standard positive force* when the signs of algebraical expressions have to be taken into account will be supposed repulsive. When the forces under consideration are gravitational we may suppose that the action is between electric masses, one positive and the other negative, whose numerical expressions are the same as those for the gravitating masses.

8. **Continuous Distribution of Mass.**—When the acting mass is continuously distributed, the distribution may exist throughout a volume, or over a surface, or along a line. Corresponding to a volume or space distribution, a surface distribution, and a line distribution, there are *three kinds of density*, a volume density denoted by ρ, a surface density by σ, and a line density by λ.

These three magnitudes may be defined as follows:—

The *volume density at a point* Q is the limit of the ratio of the mass contained by a sphere having Q as centre to the volume of the sphere when its radius is diminished without limit.

The *surface density at a point* Q, on a surface S, is the limit of the ratio of the mass contained by a sphere having Q as centre to the area of the portion of the surface S within the sphere when its radius is diminished without limit.

The *line density at a point* Q, on a line s, is the limit of the ratio of the mass on a portion of s having Q for its middle point to the length of this portion when it is diminished without limit.

If ρ be finite, the corresponding value of σ is an infinitely small quantity of the first order, and that of λ an infinitely small quantity of the second order.

In the case of gravitational forces actually existing in nature, ρ is always finite.

For electric forces, on the other hand, σ is usually finite.

In both cases we may, for mathematical purposes, suppose a fictitious surface distribution for which σ is finite.

When σ is finite, there may, of course, be an independent volume distribution for which ρ is finite.

In the case of forces actually existing in nature, λ can never be finite (Clerk Maxwell, "Electricity and Magnetism," Art. 81).

It is plain that m, λ, σ, and ρ are quantities of different kinds, each being one space-dimension lower than the preceding.

If ds denote a line element, dS a surface element, and $d\mathfrak{S}$ a volume element, the elements of mass corresponding respectively to the three kinds of distribution are given by the equations

$$dm = \lambda ds, \quad dm = \sigma dS, \quad dm = \rho d\mathfrak{S}.$$

Introducing the expression for dm into equations (1) we have, in the case of a volume distribution,

$$X = \int \rho \frac{x - \xi}{\mathrm{r}^3} d\mathfrak{S}, \quad Y = \int \rho \frac{y - \eta}{\mathrm{r}^3} d\mathfrak{S}, \quad Z = \int \rho \frac{z - \zeta}{\mathrm{r}^3} d\mathfrak{S}. \qquad (2)$$

Similar results may be obtained in like manner for a surface distribution and a line distribution.

9. **Attraction of Thin Cone at its Vertex.**—If $d\omega$ be the solid angle of the cone, and r the distance from the vertex of any point in its mass, the corresponding element of volume is $r^2 dr\, d\omega$, and the attracting force of the element is $\rho \dfrac{r^2\, dr\, d\omega}{r^2}$, that is $\rho\, dr\, d\omega$. In this case the elementary forces of attraction are all in the same direction, and, if the cone be homogeneous, the attraction at the vertex of a frustum of length l is $\rho\, d\omega \displaystyle\int_r^{r+l} dr$ or $\rho l d\omega$. This is independent of the distance of the frustum from the vertex. If the frustum extend at both sides of the vertex, the force exerted by the portion on one side is opposite in direction to that due to the other portion, and the resultant force is $\rho\,(l - l')\, d\omega$ towards the portion whose length is l.

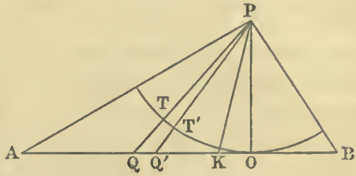
It is plain that this result holds good for any two portions of the cone which are on opposite sides of the vertex and whose lengths are l and l'.

10. **Attraction of Homogeneous Straight Line.**—The attraction of a homogeneous straight line AB at any point P is the same as that of a circular arc of equal density having P for centre, and the perpendicular distance of P from AB as radius, the extremities of the arc being on the lines PA and PB.

Let Q, Q' be any two consecutive points on AB, and p the length of the perpendicular PO on it from P, then, if PQ be denoted by r, and OPQ by θ, we have

$$QQ' = \frac{r\,d\theta}{\cos\theta},$$

and $$\frac{QQ'}{r^2} = \frac{d\theta}{p} = \frac{p\,d\theta}{p^2} = \frac{TT'}{PT^2}.$$



where T and T' are the points in which the circular arc is met by PQ and PQ'. Hence the attraction at P of QQ' is equal to that of TT', and as a similar result holds good for every element of AB, the theorem is proved.

It is now easy to find the attraction of AB at P. From symmetry, it is plain that the resultant attraction at P of the circular arc is directed along PK, the bisector of the angle APB. If we now denote KPQ by θ, we see that the component along PK of the attraction of an element of AB is expressed by $\dfrac{\lambda\cos\theta\,d\theta}{p}$. Integrating this between 0 and $\frac{1}{2}a$, where a is the angle APB, we get the component along PK of the attraction of AK; a similar result is obtained for BK; and, since the forces are both directed towards K, we have finally that the attraction of AB at P is $\dfrac{2\lambda\sin\frac{1}{2}a}{p}$, in the direction of the bisector of the angle APB.

If the length of AB be infinite, $a = \pi$, and we find that *the attraction of an infinite straight line of density* λ *at any point* P is $\dfrac{2\lambda}{p}$ in the direction of the perpendicular p from P on the line.

We shall find that this is a result of considerable importance.

11. **Cylindrical Distribution of Mass.**—When the space occupied by the acting mass is bounded by parallel cylinders of infinite length, the density being uniform along each straight line which is parallel to a generator of one of the cylinders, the distribution may be called *cylindrical.*

In this case, by Art. 10, the attraction, at any point P, of an infinitely long cylinder along whose axis the density is ρ, and whose section ε is infinitely small, is $\frac{2\rho\varepsilon}{p}$, where p is the perpendicular drawn from P to the axis of the cylinder, and, the foot of this perpendicular being Q, the direction of the attraction is the line PQ. If we draw a plane through P perpendicular to the generators of the cylinders, it contains the feet of all the perpendiculars on their axes; and if dS denote the element of this plane which is enclosed by the cylinder whose section is ε, and r the distance from P to Q, we have $\varepsilon = dS$, and $p = \mathrm{r}$, whence the force at P due to the thin cylinder is $\frac{2\rho\, dS}{\mathrm{r}}$.

Hence the resultant force at P is that due to a *uniplanar* distribution of mass in the plane through P perpendicular to the generators of the cylinders, the density τ of this distribution at a point Q being 2ρ, and the *force at P due to any element of mass varying inversely as its distance.*

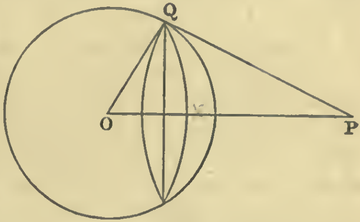
The student must not confound a uniplanar distribution of mass, such as has been described above, with a surface distribution in which the surface happens to be a plane. The definition of the uniplanar density τ is verbally the same as that of the surface density σ, but the two magnitudes are of different kinds, since in one case the force caused by an element of mass varies inversely as its distance, and in the other case inversely as the square of its distance. In fact τ is a magnitude of the same kind as ρ, the volume density.

The conditions required for a cylindrical distribution of mass may be approximately fulfilled in nature, and a uniplanar distribution is merely a *mathemathical artifice* by which the problems belonging to a cylindrical distribution may be presented in a simpler form.

12. **Attraction of Spherical Shell.**—To find the attraction of an infinitely thin homogeneous spherical shell at an external point P we may proceed as follows :—

Let O be the centre of the sphere bounding the shell, draw a plane perpendicular to PO cutting the sphere in a circle, let Q be any point on this circle, then, if a be the radius

of the sphere, and if we put $OPQ = \theta$, $POQ = \phi$, $PQ = r$, the element of the shell comprised between two consecutive planes perpendicular to OP is expressed by $2\pi a^2 \sin\phi\, d\phi$, and, since all the points of this element are at an equal distance from P, each component of attraction perpendicular to PO is equilibrated by an equal and opposite component; and the resultant attraction of the element at P is in the direction of the line PO, and is expressed by



$$\frac{2\pi\sigma a^2 \sin\phi\, d\phi}{r^2}\cos\theta,$$

where σ denotes the density of the shell.

Hence if R denote the attraction of the entire shell, we have

$$R = 2\pi\sigma a^2 \int \frac{\cos\theta \sin\phi\, d\phi}{r^2}.$$

To find the value of this integral we use the equations

$$r^2 = a^2 + c^2 - 2ac\cos\phi, \quad a^2 = r^2 + c^2 - 2cr\cos\theta,$$

where $c = PO$. Differentiating the first of these we have

$$r\,dr = ac\sin\phi\, d\phi,$$

and from the second we get

$$r\cos\theta = \frac{r^2 + c^2 - a^2}{2c};$$

substituting in the integral, we obtain

$$R = 2\pi\sigma a \int_{c-a}^{c+a} \frac{(r^2 + c^2 - a^2)\, dr}{2c^2 r^2} = \frac{4\pi\sigma a^2}{c^2}.$$

Since $4\pi\sigma a^2$ denotes the mass of the shell, we conclude, that *a thin spherical homogeneous shell produces the same attraction at an external point as if its entire mass were concentrated at the centre of the sphere.*

If the point P be inside the surface of the shell, we proceed to find the attraction in a similar manner, but in this case the limits of the integral are $a+c$ and $a-c$, and its value is zero. Hence we conclude, that *a thin spherical homogeneous shell exercises no attraction at an internal point.*

Since the attractions of thin concentric spherical shells at any point are all in the same direction, the attraction of a thick homogeneous shell comprised between two concentric spherical surfaces is the sum of the attractions of the thin shells into which it may be decomposed; hence, *a thick homogeneous shell bounded by concentric spheres has the same attraction at an external point as if the entire mass of the shell were concentrated at its centre, and has no attraction at a point inside its inner boundary.*

If the thick shell be not homogeneous, but be composed of *homogeneous layers* each of which is bounded by spheres which are all concentric, these results still hold good.

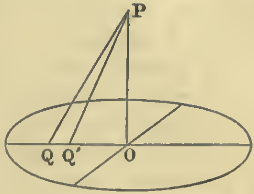
13. **Solid Sphere.**—It is plain from the last Article that a solid homogeneous sphere, or a sphere composed of homogeneous layers comprised each between spheres concentric with the outer boundary, has the same attraction at an external point as if the entire mass of the solid sphere were concentrated at its centre.

Again, *the attraction of a solid sphere at a point P in its interior is the same as that of the concentric solid sphere whose external boundary passes through P*, this theorem being equally true whether the sphere be homogeneous or be composed of homogeneous layers comprised between concentric spherical surfaces.

If M be the mass of a solid sphere S, and r the distance from its centre of the point P, the attraction at P is expressed by $\frac{M}{r^2}$ when P is external, and by $\frac{m}{r^2}$ when P is internal, m denoting the mass of the concentric sphere whose external surface passes through P. If S be homogeneous, $m = \frac{4}{3}\pi\rho r^3$, and we get for the attraction of a homogeneous sphere at a point in its interior the expression $\frac{4}{3}\pi\rho r$.

14. **Thin Plate.**—The attraction of an infinitely thin homogeneous circular plate at a point P on the perpendicular to the plane of the plate through its centre is easily found.

Let Q, and Q' be consecutive points on the same radius of the plate, and O its centre, and let $PQ = r$, $OPQ = \theta$, then $QQ' = \dfrac{r\,d\theta}{\cos\theta}$. Since all the points on the circle whose radius is OQ are equally distant from P, and the lines joining them to P make equal angles with PO, it is plain that the resultant attraction at P of the element of the plate comprised between the circles whose radii are OQ and OQ' is in the direction of the line PO, and is expressed by



$$2\pi\sigma r \sin\theta \frac{r\,d\theta}{\cos\theta}\,\frac{\cos\theta}{r^2},$$

that is, by $2\pi\sigma \sin\theta\, d\theta$. Hence the resultant attraction of the entire plate at P is in the direction PO, and if R denote its magnitude, a the radius of the plate, θ_1 the angle subtended by this radius at P, and c the distance OP, we have

$$R = 2\pi\sigma \int_0^{\theta_1} \sin\theta\, d\theta = 2\pi\sigma\left(1 - \frac{c}{\sqrt{(a^2+c^2)}}\right). \quad (3)$$

If we suppose a infinite, c remaining finite, we find that *the attraction of an infinitely thin homogeneous plate of infinite extent is constant at all external points and is expressed by* $2\pi\sigma$.

Again, if c be infinitely small compared with a, we find that *the attraction of an infinitely thin homogeneous circular plate at a point P infinitely near the plate, on the perpendicular through its centre, is in the direction of this perpendicular, and is independent of the radius of the plate*, being expressed by $2\pi\sigma$.

Hence we may conclude that Z, the component of the attraction of an infinitely thin homogeneous plate at an infinitely near point P in the direction of the perpendicular from P on the plate, is independent of the size of the plate and of the form of its bounding curve, and is expressed by $2\pi\sigma$, provided the perpendicular from P falls inside the plate.

To prove this, suppose two circles A_1 and A_2, the one outside, the other inside, the boundary of the plate, and having O the foot of the perpendicular from P as their common centre, then, if Z_1 denote the attraction of the circular plate bounded by A_1, and Z_2 of that bounded by A_2, the magnitude of Z lies between those of Z_1 and Z_2, but, as $Z_1 = Z_2 = 2\pi\sigma$, the theorem is proved.

Lastly, if the boundary of an infinitely thin homogeneous plate be such that all diameters of the plate drawn through a point O are bisected at O, it is plain that the resultant attraction of the plate, at a point P in the perpendicular to the plate through O, is in the direction OP, and hence if P approach infinitely near O, this attraction is expressed by $2\pi\sigma$.

15. **Continuity of Force in Volume Distribution.**—Equations (2), Art. 8, which give the components of the force at a point P, if we use polar coordinates and take P for origin, become, when the acting mass is attractive,

$$X = \int \rho \sin^2\theta \cos\phi \, dr\, d\theta\, d\phi, \quad Y = \int \rho \sin^2\theta \sin\phi \, dr\, d\theta\, d\phi,$$

$$Z = \int \rho \cos\theta \sin\theta \, dr\, d\theta\, d\phi.$$

The integrations here indicated remain valid when P is inside the acting mass, and if we integrate with respect to r between O and an infinitely small value a, we see that the mass infinitely near P cannot contribute more than an infinitely small quantity of the order a to the attraction-components. Hence, if P receive a small displacement less than a, the corresponding changes in X, Y, Z, due to mass infinitely near P must be of the same order; it is plain that this also must be the order of the changes in X, Y, Z due to mass at a distance from P greater than a. Hence, X, Y, Z in the case of a volume distribution remain finite and continuous when P is inside the acting mass.

16. **Discontinuity of Force in Surface Distribution.**—In the case of a distribution on the surface S the force components at a point P vary continuously so long as P is at a finite distance from S.

If O be the point in which the normal through P meets the surface, when P is distant from O by an infinitely small

quantity δ of the second order the points of the surface whose distances from P are less than an infinitely small quantity a of the first order lie inside a sphere whose equation referred to O as origin is $x^2 + y^2 + (z - \delta)^2 = a^2$, the normal at O being taken as the axis of z; also the equation of the surface is of the form $z = u_2 +$ &c., and the points of the surface whose distances from P are of the first order lie on the surface $z = u_2$, where u_2 is a quadratic function of x and y. The equation of the projection on the tangent plane at O of the curve of intersection of this surface and the sphere is

$$x^2 + y^2 + (u_2 - \delta)^2 = a^2.$$

Infinitely small quantities of an order higher than a^2 being neglected, this becomes $x^2 + y^2 = a^2$, which represents a circle of radius a.

The projection on the tangent plane at O of an element of surface dS is $dS \cos \psi$, where ψ is the angle which the normal to the element dS makes with OP. When dS and P are infinitely near, $\cos \psi$ differs from unity by an infinitely small quantity of the second order, also the distances from P of the surface element and its projection differ by a quantity infinitely small as compared with these distances, and finally the difference of the angles which these distances make with PO is infinitely small compared with the angles themselves.

Hence, if their densities be the same, the force exerted at P by the elements of the surface S at a distance from P less than a is the same as that due to their projections on the tangent plane at O, that is, the same as that due to the circular plate whose centre is O, whose radius is a, and whose density is σ the surface density at O.

We conclude therefore, by Art. 14, that the surface mass infinitely near P exercises no attraction at P in the direction of a tangent at O, but produces a force in the direction of the normal whose magnitude is expressed by $2\pi\sigma$. If the mass be repulsive, this force is away from the surface on whichever side of it P be situated.

It appears then from what has been said that as P approaches and passes through the surface S at the point O the tangential components at P of the repulsion of the mass

infinitely near O are continuously zero, but the normal component changes from $-2\pi\sigma$ to $+2\pi\sigma$, that is, its value in the direction in which P is moving is increased by $4\pi\sigma$ as P passes through the surface S.

The force components due to mass at a distance from P greater than a are obviously continuous. Hence on the whole, when the acting mass is repulsive, the force component along the normal in the direction in which P is moving increases by $4\pi\sigma$ as P passes through a surface on which mass of density σ is distributed, but the other force components are not altered by any finite amount. It is plain that in all cases, whether P be on the surface or not, all the force components remain finite.

The algebraical expressions for X, Y, Z as integrals derived from equations (1), Art. 7, are not in general valid, in the case of a surface distribution, for a point on the surface on which the mass is distributed, as the quantities under the integral sign may in this case become infinite within the limits of the integration.

17. **Magnets.**—The leading characteristic of magnetic forces is the presence in each element of the magnetized body of two centres of force of equal and opposite intensities, these centres of force being situated at the extremities of an infinitely short line whose direction is that of the magnetization of the element. The two centres of force are called poles, one being a north pole, the other a south pole. Each repels a pole of like kind to itself and attracts one of the opposite kind. The direction of magnetization is reckoned from the south pole to the north. The terms south and north are used in reference to the Earth's action on the magnet, the pole which is attracted towards the north being called the north pole.

When a line of any form is *uniformly magnetized* in the direction of its length, the effect in external space of the north pole at the end of one element is neutralized by that of the south pole at the beginning of the consecutive element. Thus the effect of such a magnet on a north pole of unit intensity in external space is entirely due to the south pole at one of its extremities and the north pole at the other. These centres of force are called the *poles of the magnet*. A magnet

whose length is considerable compared with its other dimensions, and which is magnetized in the direction of its length may be looked upon as linear. It may be straight or be bent into any other shape. Very small magnets may be regarded as magnetic particles. To such particles magnetized iron filings are an approximation.

The theory of magnetized bodies having more than one finite dimension must be reserved for a future chapter.

The force which a magnet pole exerts on a given pole at a given distance is proportional to its own magnetic intensity or strength. Hence, by properly selecting the units employed, we may define the *strength of a magnet pole* as the force which it exerts on the unit pole at the unit distance. The magnetization of a north pole is counted positive, and that of a south pole negative.

In the case of magnetic particles, or of linear magnets uniformly magnetized in the direction of their length, the product of the strength of a pole and of the distance between the poles is called the *magnetic moment of the magnet.*

The straight line joining the poles is called the *axis of the magnet* and the middle point of this line the *centre of the magnet.*

EXAMPLES.

1. If β be the intercept on the bisector of the vertical angle of a triangle ACB made by a perpendicular let fall from the middle point of the base AB, show that the attraction of AB at C may be expressed in the form

$$\frac{8\lambda c\beta}{(a+b)\{(a+b)^2-c^2\}}$$

where a, b, c are the sides of the triangle, and λ the density of c. (Thomson and Tait.)

Let CK be the bisector of the angle ACB, and H the foot of the perpendicular let fall on it from M the middle point of AB. Draw AI and BJ perpendicular to CK, then

$$IJ = 2HI, \text{ and } \beta = CH = \tfrac{1}{2}(CI + CJ) = \tfrac{1}{2}(a+b)\cos\tfrac{1}{2}C.$$

If R denote the attraction of AB at C, we have, Art. 10,

$$R = \frac{2\lambda \sin\frac{1}{2}C}{p} = \frac{2\lambda c\sin\frac{1}{2}C}{ab\sin C} = \frac{\lambda c(a+b)\cos\frac{1}{2}C}{ab(a+b)\cos^2\frac{1}{2}C} = \frac{\lambda c(a+b)\cos\frac{1}{2}C}{(a+b)\,s(s-c)}$$

$$= \frac{8\lambda c\,\frac{1}{2}(a+b)\cos\frac{1}{2}C}{(a+b)\,2s\,2(s-c)} = \frac{8\lambda c\beta}{(a+b)\{(a+b)^2-c^2\}}.$$

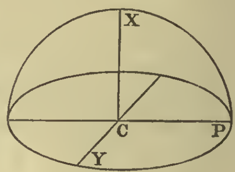
2. The base AB of a triangle ABC extends both ways to infinity, the part beyond A repelling, and that beyond B attracting with a force varying inversely as the square of the distance, the density being uniform: show that the direction of the resultant force at C is that of the bisector of the external vertical angle of the triangle.

3. Find the attraction at the vertex of a right circular cone of a frustum of the cone intercepted between two planes perpendicular to the axis.

If r be the distance of any point Q on the axis from the vertex O, the attraction at O of an infinitely thin frustum of the cone passing through Q by Art. 14 is $2\pi\rho dr\,(1 - \cos\alpha)$, where α is the semi-angle of the cone.

Hence, if R be the attraction of a frustum whose length along the axis of the cone is l, we have $R = 2\pi\rho l(1 - \cos\alpha)$.

4. Find the attraction of a solid homogeneous hemisphere at a point P on its edge.



Let a denote the radius of the hemisphere, and let C be its centre. Draw a plane through PC perpendicular to the base, the resultant attraction lies in this plane; let Z denote its component along PC, and X perpendicular to PC, then, if r be the radius vector from P to the surface of the hemisphere, we have

$$Z = 2\rho \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r\cos\theta\,\sin\theta\,d\theta\,d\phi = 4\rho a \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos^2\theta\,\sin\theta\,d\theta\,d\phi = \frac{2\pi\rho a}{3}.$$

$$X = 4\rho a \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos\theta\,\sin^2\theta\,\cos\phi\,d\theta\,d\phi = \frac{4\rho a}{3}.$$

5. Show that the *observed* latitudes of stations due north and south of a hemispherical hill of radius a and density ρ differ by $\frac{a}{R}\left(2 + \frac{\rho}{\epsilon}\right)$, where R denotes the radius and ϵ the mean density of the Earth. (Thomson and Tait.)

6. Find the attraction of a homogeneous rectangular parallelepiped, one of whose edges is infinitely long, at a point P on this edge.

Take P for origin, and the infinitely long edge as axis of z, then, if X and Y denote the components of attraction parallel to the edges whose lengths are a and b, and ϕ_1 the angle whose tangent is $\frac{a}{b}$, we have

$$X = \rho a \int_0^{\pi} \int_{\phi_1}^{\frac{1}{2}\pi} \sin\theta\,d\theta\,d\phi + \rho b \int_0^{\pi} \int_0^{\phi_1} \sin\theta\,\tan\phi\,d\theta\,d\phi.$$

Hence
$$X = \rho a \left\{\pi - 2\tan^{-1}\frac{a}{b}\right\} + 2\rho b \log \frac{\sqrt{a^2 + b^2}}{b}.$$

In like manner

$$Y = \rho b \left\{\pi - 2\tan^{-1}\frac{b}{a}\right\} + 2\rho a \log \frac{\sqrt{a^2 + b^2}}{a}.$$

7. Show that a deep narrow crevasse of great length extending east and west increases the apparent latitude of places at its southern edge by the angle $\frac{3}{4}\frac{\rho a}{\epsilon R}$ approximately, where ϵ is the mean density of the earth, ρ the density of its crust, R its radius, and a the breadth of the crevasse. (Thomson and Tait.)

8. Prove that the component of the attraction of a straight line AB of density λ parallel to itself in the direction from B to A at a point P is

$$\frac{\lambda}{PB} - \frac{\lambda}{PA}.$$

9. Show that, if the point P be in the production of AB, the attraction component perpendicular to AB is zero, and that if P lie between A and B this component is infinite if λ be finite. Show also that for both these positions of P the expression in Question 8 for the attraction component parallel to AB holds good.

10. In the case of mass acting inversely as the distance, prove that, if two curves be inverse to each other relatively to a point O, corresponding elements of the two curves whose densities are equal produce the same attraction at O.

Here the attraction at O of the element $ds = \frac{\lambda ds}{r} = \frac{\lambda r d\theta}{r \sin\phi} = \frac{\lambda d\theta}{\sin\phi}$, where ϕ is the angle which the radius vector from O makes with the tangent to the element ds. In like manner, attraction of $ds' = \frac{\lambda d\theta}{\sin\phi'}$, but in the case of inverse curves $\phi' = \pi - \phi$; $\therefore$ &c.

11. In the case of mass acting inversely as the square of the distance, prove that corresponding elements of the two curves have the same attraction at O if the densities of the elements be proportional to the perpendiculars from O on their tangents.

12. Prove that the attraction of a right circular cylinder at a point P on its axis is $2\pi\rho\,\{z_2 - z_1 - (r_2 - r_1)\}$, where ρ is the density of the cylinder, z_2 and z_1 the distances from P of the two ends of its axis, and r_2 and r_1 the distances of points on the boundaries of the terminal plane faces.

13. If g' denote the acceleration due to gravity on the top of a table-land of height h, and g its value on the mean surface of the Earth in the neighbourhood of the table-land, show that

$$g' = g\left\{1 - \left(2 - \frac{3\rho}{2\epsilon}\right)\frac{h}{a}\right\},$$

where a denotes the radius of the Earth, ϵ its mean density, and ρ the density of the table-land.

The table-land may be regarded as a thin circular plate of surface-density ρh, also $g = \frac{4}{3}\pi\epsilon a$, and at the top of the table-land the attraction of the rest of the Earth is $\frac{ga^2}{(a+h)^2}$. Hence, by addition, g' is obtained.

14. A triangle is formed of three uniform bars whose densities are λ, μ, ν, and whose elements attract with a force varying inversely as the square of the distance; determine the point O inside the triangle at which a particle would be in equilibrium.

If p, q, r be the perpendiculars from the point O on the three sides of the triangle, the position of O is determined by the equations

$$\frac{\lambda}{p} = \frac{\mu}{q} = \frac{\nu}{r}.$$

If $\lambda = \mu = \nu$ the point O is the centre of the circle inscribed in the triangle.

15. A uniform bar attached to smooth hinges at its extremities, A and B, is attracted by a force directed to a fixed point C, and varying inversely as the square of the distance from C. If P be any point on the bar, and the angle CPA be denoted by ϕ, prove that the bending moment M at P is given by the equation

$$M = \mu \frac{\sin(A+\phi)}{\sin\phi}\left\{\cot\tfrac{1}{2}(A+B) - \cot\tfrac{1}{2}(A+\phi)\right\},$$

where μ is the intensity of the attractive force per unit of length.

The resultant attraction F on PA is in the direction of CQ the bisector of the angle PCA, and its magnitude is $\frac{2\mu}{p}\sin\frac{PCA}{2}$, where p is the perpendicular from C on AB. The moment of F round P is

$$\frac{2\mu\, AP \sin A}{p}\,\frac{\cos^2\frac{1}{2}(A+\phi)}{\sin(A+\phi)},\quad \text{that is,}\quad \frac{\mu AP}{AC}\cot\tfrac{1}{2}(A+\phi).$$

If P coincide with B this moment must be equal and opposite to the moment round B of the force perpendicular to AB exerted by the hinge at A. Hence, if R denote this force,

$$R\,.\,AB = \mu\frac{AB}{AC}\cot\tfrac{1}{2}(A+B),$$

and the moment of R round P is

$$\frac{\mu AP}{AC}\cot\tfrac{1}{2}(A+B).$$

We have, then,

$$M = \frac{\mu AP}{AC}\left\{\cot\tfrac{1}{2}(A+B) - \cot\tfrac{1}{2}(A+\phi)\right\}$$

$$= \mu\frac{\sin(A+\phi)}{\sin\phi}\left\{\cot\tfrac{1}{2}(A+B) - \cot\tfrac{1}{2}(A+\phi)\right\}.$$

16. In the last example determine where the bending moment M is a maximum.

From the expression for M given above we readily obtain

$$M = 2\mu\frac{\cos\frac{1}{2}(A+\phi)\sin\frac{1}{2}(\phi-B)}{\sin\frac{1}{2}(A+B)\sin\phi}$$

$$= \frac{\mu}{\sin\frac{1}{2}(A+B)}\,\frac{\sin\{\phi+\frac{1}{2}(A-B)\} - \sin\frac{1}{2}(A+B)}{\sin\phi}.$$

When this is a maximum

$$\sin\phi\cos\{\phi+\tfrac{1}{2}(A-B)\}-\cos\phi\,\{\sin[\phi+\tfrac{1}{2}(A-B)]-\sin\tfrac{1}{2}(A+B)\}=0,$$

that is, $\cos\phi\sin\frac{1}{2}(A+B)=\sin\frac{1}{2}(A-B)$, which determines ϕ, and therefore P.

17. Iron filings are sprinkled on a piece of paper which is laid over the two poles of a horse-shoe magnet: find the curve traced out by a continuous set of filings.

Any filing PQ becomes a small magnet, each of whose extremities is acted on by two forces, one attractive, and the other repulsive. For the equilibrium of the filing the resultant forces passing through P and Q must be equal and opposite, whence PQ must be in the direction of the resultant force at Q. Let S and N be the poles of the magnet, and L any point on the production of SN, let $NQ=r$, $Sq=r'$, $QNL=\theta$, $QSN=\theta'$; then, if m be the strength of the pole at S or N, and ds the element of the required curve, we have

$$\frac{m}{r^2}:\frac{m}{r'^2}=\frac{r'\,d\theta'}{ds}:\frac{r\,d\theta}{ds};$$

$$\therefore\quad \frac{d\theta}{d\theta'}=\frac{r}{r'}=\frac{\sin\theta'}{\sin\theta},\ \text{whence}\ \cos\theta'-\cos\theta=C,$$

which determines the curve, C being an arbitrary constant.

Curves traced out in the manner described above are called *magnetic curves.*

18. In the last example show that filings whose lines of direction pass through the same point O on SN lie on a circle.

Since $$\frac{m}{r^2}:\frac{m}{r'^2}=\sin SQO:\sin NQO,$$

we have $$\frac{r'^2}{r^2}=\frac{SO}{NO}\frac{r}{r'};\quad\therefore\ \frac{r'}{r}=\left(\frac{SO}{NO}\right)^{\frac{1}{3}}=\text{constant},$$

whence the locus of Q is a circle.

19. Show that at any point P the component of the attraction of a homogeneous plane lamina in the direction perpendicular to its plane is expressed by $\sigma\Omega$, where σ is the density of the lamina, and Ω the solid angle which it subtends at P.

If ψ be the angle which a radius vector r from P to a point in the lamina makes with the perpendicular to its plane, an element of the lamina is expressed by $\dfrac{r^2\,d\omega}{\cos\psi}$, and the component of its attraction at P by $\sigma\,d\omega$: $\therefore$ &c.

20. Prove that the attraction of a homogeneous elliptic plate at a point P on the perpendicular through its centre is expressed by

$$2\pi\sigma-\frac{4\sigma bz}{a\sqrt{(a^2+z^2)}}\int_0^{\frac{\pi}{2}}\frac{d\psi}{(1-e^2\sin^2\psi)\sqrt{(1-\kappa^2\sin^2\psi)}},$$

where z is the distance of P from the centre of the ellipse, a and b its semiaxes, e its eccentricity, and $\kappa^2 = \dfrac{a^2 - b^2}{a^2 + z^2}$.

Here $$\Omega = \iint \sin\theta \, d\theta \, d\phi = \int (1 - \cos\theta) \, d\phi = 2\pi - 4\int_0^{\frac{\pi}{2}} \cos\theta \, d\phi.$$

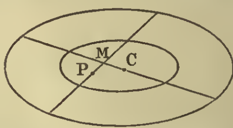
If ϖ be the central radius vector of the ellipse, and if we put

$$\varpi \cos\phi = a \cos\psi, \quad \varpi \sin\phi = b \sin\psi,$$

we have $$a^2 \cos^2\psi + b^2 \sin^2\psi = \varpi^2 = z^2 \tan^2\theta,$$

and the limiting values of ψ are the same as those of ϕ. Hence by substitution we have the result above.

18. **Ellipsoidal Shell.**—A homogeneous shell bounded by similar, concentric, and coaxal ellipsoids, is called a *thick homœoid.* Such a shell exercises no attraction at a point P inside its internal surface.



To prove this, suppose a cone of infinitely small angle, having its vertex at P, and extending in both directions to the outer boundary of the shell. The ellipsoids being similar, the plane CM conjugate to the axis of this cone is the same for the outer and the inner ellipsoid, and therefore the intercepts on this axis on opposite sides of P between the inner and outer boundaries of the shell are equal. Hence, by Art. 9, the mass inside the cone exercises no attraction at P, but by supposing an infinite number of such cones the whole mass of the shell may be exhausted; hence the total attraction of the shell at P is zero.

It is plain that we can show in a similar manner that, in the case of a uniplanar distribution of mass attracting inversely as the distance, a homogeneous band bounded by similar concentric and coaxal ellipses exercises no attraction at a point inside its inner boundary.

A spherical shell, whose attraction at an internal point has been investigated in Art. 12, is obviously a particular case of an ellipsoidal shell bounded by similar ellipsoids. When the external and internal surfaces of a homœoidal shell approach infinitely near to each other, the shell is called simply a *homœoid.*

19. **Elliptic Plate.**—The attraction of a homogeneous elliptic plate at a point on the extremity of an axis, for a uniplanar distribution of mass attracting inversely as the distance, may be found as follows:—

Let $2a$ and $2b$ denote the axes of the ellipse, and Y_2 the component in the direction of the axis of y of the attraction of the plate at the point B whose coordinates referred to the centre are $0, -b$. Taking B for origin, the equation of the ellipse referred to polar coordinates is

$$r = \frac{2a^2 b \sin\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta}.$$

The attraction of a triangle, of infinitely small angle $d\theta$, having B for vertex, and extending across the ellipse, is expressed by $\tau r\, d\theta$. Hence

$$Y_2 = 2\tau \int_0^{\frac{\pi}{2}} r \sin\theta\, d\theta.$$

Substituting for r from the equation of the ellipse, we have

$$Y_2 = 4\tau a^2 b \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\, d\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta}$$

$$= \frac{4\tau a^2 b}{a^2 - b^2} \int_0^{\frac{\pi}{2}} \left(d\theta - \frac{\sec^2\theta\, d\theta}{1 + \frac{a^2}{b^2} \tan^2\theta} \right)$$

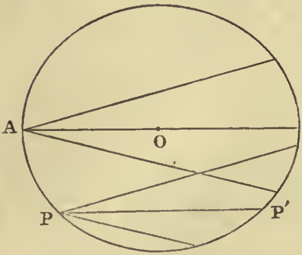
$$= \frac{4\tau a^2 b}{a^2 - b^2} \left(\frac{\pi}{2} - \frac{b}{a} \frac{\pi}{2} \right) = \frac{2\pi\tau ab}{a + b}.$$

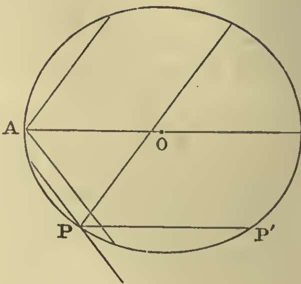
Since an ellipse is symmetrical on each side of an axis it is plain that the attraction component at B parallel to the axis major is zero. If X_1 be the attraction of the plate at an extremity of the axis major, we find, by a similar process, that X_1 is codirectional with the axis major, and is equal to Y_2.

We can now prove that at a point P on the boundary, whose coordinates referred to the centre are $-x$ and $-y$, the

components, X and Y, parallel to the axes, of the attraction of the plate are given by the equations

$$X = \frac{x}{a} X_1, \quad Y = \frac{y}{b} Y_2.$$





To show this, draw through P a parallel PP' to the axis major AO. Let R_1 and R_2 denote two chords of the ellipse drawn through the extremity A of the axis major, and making equal angles with it on opposite sides, r_1 and r_2 the parallel chords through P, and θ_1 and θ_2 the angles they make with PP'; then $\theta_2 = -\theta_1$ when r_1 and r_2 are on opposite sides of PP', and $\theta_2 = \pi - \theta_1$ when they are on the same side. Again r, a chord drawn through P, is given by the equation

$$r\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 2\left(\frac{x\cos\theta}{a^2} + \frac{y\sin\theta}{b^2}\right),$$

and R, the parallel chord through A by the equation

$$R\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = \pm\,\frac{2a\cos\theta}{a^2};$$

hence $$r_1\cos\theta_1 + r_2\cos\theta_2 = \frac{x}{a}(R_1 + R_2)\cos\theta_1.$$

Also the attraction component along PP' of the elementary triangle whose side is r is expressed by $\tau r\cos\theta\, d\theta$.

Hence we have

$$X = \tau \int_0^{\frac{\pi}{2}} (r_1 \cos\theta_1 + r_2 \cos\theta_2)\, d\theta_1 = \tau \frac{x}{a} \int_0^{\frac{\pi}{2}} (R_1 + R_2) \cos\theta_1\, d\theta_1$$

$$= \frac{x}{a} X_1, \quad \text{similarly} \quad Y = \frac{y}{b} Y_2.$$

If we substitute for X_1 and Y_2 the values already obtained, we get

$$X = \frac{2\pi\tau bx}{a+b} = \frac{2M}{a+b}\frac{x}{a}, \qquad Y = \frac{2\pi\tau ay}{a+b} = \frac{2M}{a+b}\frac{y}{b}, \qquad (4)$$

where M is the uniplanar mass of the plate.

If P be inside the ellipse bounding the plate, suppose a similar concentric ellipse, whose axes are $2a'$ and $2b'$, drawn through P, then the band between this ellipse and the boundary, by Art. 18, exercises no attraction at P, and the attraction components X and Y of the plate are given by the equations

$$X = \frac{2\pi\tau b'x}{a'+b'} = \frac{2\pi\tau bx}{a+b}, \quad Y = \frac{2\pi\tau a'y}{a'+b'} = \frac{2\pi\tau ay}{a+b}. \qquad (5)$$

20. **Infinite Elliptic Cylinder.**—If we put $\tau = 2\rho$ in equations (5), we find, by Art. 11, that the attraction components X and Y of an infinitely long homogeneous elliptic cylinder, whose axis is the axis of z, at a point $-x$, $-y$, on or inside the bounding surface, are given by the equations

$$X = \frac{4\pi\rho bx}{a+b}, \quad Y = \frac{4\pi\rho ay}{a+b}. \qquad (6)$$

21. **Ellipsoid.**—The attraction of a homogeneous ellipsoid at a point on its surface may, in a manner similar to that employed in Article 19, be deduced from the attraction at a point P at an extremity of an axis.

To find this latter attraction, let the axes of the ellipsoid be denoted by $2a$, $2b$, $2c$, and let P be the point whose coordinates referred to the centre are 0, 0, $-c$, then the equation of the ellipsoid in polar coordinates referred to P as origin is

$$r\left(\frac{\sin^2\theta\cos^2\phi}{a^2} + \frac{\sin^2\theta\sin^2\phi}{b^2} + \frac{\cos^2\theta}{c^2}\right) = \frac{2\cos\theta}{c}.$$

The attraction at P of a cone of infinitely small angle having its vertex at P, and extending across the ellipsoid, is expressed by $\rho r \sin\theta \, d\theta \, d\phi$. Hence, if Z_3 denote the attraction component of the ellipsoid at P in the direction of the axis of z, we have

$$Z_3 = \rho \int_0^{\frac{\pi}{2}} \int_0^{2\pi} r \sin\theta \cos\theta \, d\theta \, d\phi = 4\rho \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r \sin\theta \cos\theta \, d\theta \, d\phi.$$

Substituting for r from the equation of the ellipsoid, we obtain

$$Z_3 = 8\rho a^2 b^2 c \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta \sin\theta \, d\theta \, d\phi}{a^2 b^2 \cos^2\theta + c^2 \sin^2\theta \, (a^2 \sin^2\phi + b^2 \cos^2\phi)}.$$

Put $\quad a^2 b^2 \cos^2\theta = C, \quad a^2 c^2 \sin^2\theta = A, \quad b^2 c^2 \sin^2\theta = B,$

$$\text{then,} \quad Z_3 = 8\rho a^2 b^2 c \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, d\theta \int_0^{\frac{\pi}{2}} \frac{d\phi}{C + A \sin^2\phi + B \cos^2\phi}.$$

If we integrate with respect to ϕ we get for the corresponding integral

$$\frac{1}{\sqrt{(C+A)(C+B)}} \tan^{-1}\left(\sqrt{\frac{C+A}{C+B}} \tan\phi\right);$$

this between the assigned limits is

$$\frac{\pi}{2\sqrt{(C+A)(C+B)}};$$

hence

$$Z_3 = 4\pi\rho a^2 b^2 c \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta \sin\theta \, d\theta}{\sqrt{(a^2 b^2 \cos^2\theta + b^2 c^2 \sin^2\theta)(a^2 b^2 \cos^2\theta + a^2 c^2 \sin^2\theta)}}. \tag{7}$$

If we put $\cos\theta = u$, we get finally,

$$Z_3 = 4\pi\rho abc \int_0^1 \frac{u^2 du}{[c^2 + (a^2 - c^2) u^2]^{\frac{1}{2}} [c^2 + (b^2 - c^2) u^2]^{\frac{1}{2}}}. \tag{8}$$

From the symmetry of the ellipsoid round its axes it is plain that its other attraction components at P are zero, so

that Z_3 denotes the total resultant attraction at the extremity of the semi-axis c.

We obtain, in like manner,

$$X_1 = 4\pi\rho abc \int_0^1 \frac{u^2\,du}{\{a^2 + (b^2 - a^2)\,u^2\}^{\frac{1}{2}}\,\{a^2 + (c^2 - a^2)\,u^2\}^{\frac{1}{2}}}, \qquad (9)$$

$$Y_2 = 4\pi\rho abc \int_0^1 \frac{u^2\,du}{\{b^2 + (c^2 - b^2)\,u^2\}^{\frac{1}{2}}\,\{b^2 + (a^2 - b^2)\,u^2\}^{\frac{1}{2}}}, \qquad (10)$$

where X_1 and Y_2 denote the attractions at the extremities of the semi-axes a and b. The expression for Z_3 in equation (8) can be put into a more convenient form by taking the factor c^2 in the denominator outside the integral sign, and by putting

$$\lambda_1^2 = \frac{a^2 - c^2}{c^2}, \quad \lambda_2^2 = \frac{b^2 - c^2}{c^2},$$

we have, then,

$$Z_3 = \frac{4\pi\rho ab}{c} \int_0^1 \frac{u^2\,du}{(1 + \lambda_1^2 u^2)^{\frac{1}{2}}\,(1 + \lambda_2^2 u^2)^{\frac{1}{2}}}. \qquad (11)$$

To make X_1 depend on λ_1 and λ_2, assume

$$\frac{a^2}{u^2} - a^2 = \frac{c^2}{v^2} - c^2, \quad \text{then} \quad du = \frac{c^2}{a^2}\left(\frac{u}{v}\right)^3 dv,$$

and

$$\left(\frac{u}{v}\right)^2 = \frac{a^2}{c^2\,(1 + \lambda_1^2 v^2)}.$$

Substituting for u in terms of v in equation (9), since the limiting values of v are the same as those of u, we get

$$X_1 = \frac{4\pi\rho ab}{c}\,\frac{a}{c} \int_0^1 \frac{v^2\,dv}{(1 + \lambda_1^2 v^2)^{\frac{3}{2}}\,(1 + \lambda_2^2 v^2)^{\frac{1}{2}}}. \qquad (12)$$

If a and b be interchanged, X_1 and λ_1 become Y_2 and λ_2; hence we obtain

$$Y_2 = \frac{4\pi\rho ab}{c}\,\frac{b}{c} \int_0^1 \frac{v^2\,dv}{(1 + \lambda_1^2 v^2)^{\frac{1}{2}}\,(1 + \lambda_2^2 v^2)^{\frac{3}{2}}}. \qquad (13)$$

If we put $M = \frac{4}{3}\pi\rho abc$, then M will denote the mass of the ellipsoid, and X_1, Y_2, Z_3 may be expressed by the equations

$$\left.\begin{aligned} X_1 &= \frac{3M}{c^3}\, a \int_0^1 \frac{u^2\,du}{(1+\lambda_1^2u^2)^{\frac{3}{2}}\,(1+\lambda_2^2u^2)^{\frac{1}{2}}}, \\ Y_2 &= \frac{3M}{c^3}\, b \int_0^1 \frac{u^2\,du}{(1+\lambda_1^2u^2)^{\frac{1}{2}}\,(1+\lambda_2^2u^2)^{\frac{3}{2}}}, \\ Z_3 &= \frac{3M}{c^3}\, c \int_0^1 \frac{u^2\,du}{(1+\lambda_1^2u^2)^{\frac{1}{2}}\,(1+\lambda_2^2u^2)^{\frac{1}{2}}}, \end{aligned}\right\}. \qquad (14)$$

If now X, Y, Z denote the components parallel to the axes of the attraction of an ellipsoid at any point P on its external surface, whose coordinates are $-x$, $-y$, $-z$, it is easy to show that

$$X = \frac{x}{a} X_1, \quad Y = \frac{y}{b} Y_2, \quad Z = \frac{z}{c} Z_3.$$

To prove this, draw through P a parallel PP' to CO, where C is the extremity of the semi-axis c, and O is the centre of the ellipsoid, let R_1 and R_2 denote two chords of the ellipsoid lying in a plane passing through CO and making equal angles with it on opposite sides at the point C, and let r_1 and r_2 denote the parallel chords through P, and θ_1 and θ_2 the angles they make with PP'. Then $\theta_2 = -\theta_1$ when r_1 and r_2 are on opposite sides of PP', and $\theta_2 = \pi - \theta_1$ when they are on the same side; also R the chord drawn through C is given by the equation

$$R\left(\frac{\sin^2\theta\cos^2\phi}{a^2} + \frac{\sin^2\theta\sin^2\phi}{b^2} + \frac{\cos^2\theta}{c^2}\right) = \pm\frac{2c\cos\theta}{c^2},$$

and r the parallel chord through P by the equation

$$r\left(\frac{\sin^2\theta\cos^2\phi}{a^2} + \frac{\sin^2\theta\sin^2\phi}{b^2} + \frac{\cos^2\theta}{c^2}\right)$$
$$= \frac{2x\sin\theta\cos\phi}{a^2} + \frac{2y\sin\theta\sin\phi}{b^2} + \frac{2z\cos\theta}{c^2};$$

hence $\qquad r_1\cos\theta_1 + r_2\cos\theta_2 = \frac{z}{c}(R_1 + R_2)\cos\theta_1.$

Now $$Z = \rho \int_0^{\frac{\pi}{2}} \int_0^{\pi} (r_1 \cos\theta_1 + r_2 \cos\theta_2) \sin\theta_1 \, d\theta_1 \, d\phi,$$

and $$Z_3 = \rho \int_0^{\frac{\pi}{2}} \int_0^{\pi} (R_1 + R_2) \cos\theta_1 \sin\theta_1 \, d\theta_1 \, d\phi,$$

whence $Z = \dfrac{z}{c} Z_3$, and similarly, $X = \dfrac{x}{a} X_1$ and $Y = \dfrac{y}{b} Y_2$.

A purely geometrical method of arriving at these equations will be found in Ex. 12, Art. 24.

If we substitute for X_1, Y_2, and Z_3, their values obtained from equations (14), we have

$$\left.\begin{aligned} X = Ax &= \frac{3M}{c^3}\, x \int_0^1 \frac{u^2\,du}{(1+\lambda_1^2 u^2)^{\frac{3}{2}} (1+\lambda_2^2 u^2)^{\frac{1}{2}}}, \\ Y = By &= \frac{3M}{c^3}\, y \int_0^1 \frac{u^2\,du}{(1+\lambda_1^2 u^2)^{\frac{1}{2}} (1+\lambda_2^2 u^2)^{\frac{3}{2}}}, \\ Z = Cz &= \frac{3M}{c^3}\, z \int_0^1 \frac{u^2\,du}{(1+\lambda_1^2 u^2)^{\frac{1}{2}} (1+\lambda_2^2 u^2)^{\frac{1}{2}}}, \end{aligned}\right\} \qquad (15)$$

where A, B, C are constants defined by the equations above.

The expressions in equations (15) for the attraction components of the ellipsoid at a point P whose coordinates referred to the centre are $-x$, $-y$, $-z$, hold good if P be inside the external surface of the ellipsoid.

To prove this, suppose a similar ellipsoid, whose axes are $2a'$, $2b'$, $2c'$, drawn through P; then, by Art. 18, the total attraction at P is the attraction of this ellipsoid. Also, if M' be its mass,

$$\frac{M'}{c'^3} = \frac{M}{c^3}, \quad \text{and} \quad \lambda'_1 = \lambda_1, \ \lambda'_2 = \lambda_2;$$

therefore the coefficients of x, y, z in the expressions for the attraction components are the same whether P be on the surface of the ellipsoid or in its interior.

22. **Symmetrical Expressions for Components of Attraction.**—Symmetrical expressions for X, Y, and Z may be obtained in the following manner:—

Assume $v = c^2 \tan^2\theta$, then

$$dv = 2c^2 \sin\theta \sec^3\theta \, d\theta = 2c^2 \sin\theta \, c^{-3} (c^2 + v)^{\frac{3}{2}} \, d\theta,$$

whence $\sin\theta\, d\theta = \dfrac{c\,dv}{2\,(c^2+v)^{\frac{3}{2}}}$, also $v = \infty$ when $\theta = \dfrac{\pi}{2}$, and $v = 0$ when $\theta = 0$. Thus (7) becomes

$$Z_3 = 2\pi\rho abc^2 \int_0^\infty \frac{dv}{(a^2+v)^{\frac{1}{2}}\,(b^2+v)^{\frac{1}{2}}(c^2+v)^{\frac{3}{2}}}. \tag{16}$$

From (16) and the corresponding equations we get

$$\left.\begin{aligned}
X &= 2\pi\rho abcx \int_0^\infty \frac{dv}{(a^2+v)^{\frac{3}{2}}\,(b^2+v)^{\frac{1}{2}}\,(c^2+v)^{\frac{1}{2}}},\\
Y &= 2\pi\rho abcy \int_0^\infty \frac{dv}{(a^2+v)^{\frac{1}{2}}\,(b^2+v)^{\frac{3}{2}}\,(c^2+v)^{\frac{1}{2}}},\\
Z &= 2\pi\rho abcz \int_0^\infty \frac{dv}{(a^2+v)^{\frac{1}{2}}\,(b^2+v)^{\frac{1}{2}}\,(c^2+v)^{\frac{3}{2}}},
\end{aligned}\right\}. \tag{17}$$

By assuming $v = \mu^2 u$ it is easy to show that these expressions hold good for any point P whose coordinates are $-x$, $-y$, and $-z$, and which lies inside the ellipsoid on the surface of the similar ellipsoid whose semiaxes are μa, μb, and μc.

23. **Components of Attraction at Internal Point found directly.**—If the origin be taken at a point P inside an ellipsoid, the equation of its surface gives a quadratic equation to determine the intercepts on a radius vector r, making angles α, β, γ with the axes. If the lengths of these intercepts be denoted by r_1 and r_2, the component X of the attraction of the ellipsoid at P is given by the equation $X = \int \rho\,(r_1 - r_2)\cos\alpha\, d\omega$; but r_1 and $-r_2$ are the roots of the quadratic equation in r, whence, if $-x$, $-y$, $-z$ denote the coordinates of P referred to the centre as origin, we have

$$X = 2\rho \int \frac{\dfrac{x}{a^2}\cos^2\alpha + \dfrac{y}{b^2}\cos\alpha\cos\beta + \dfrac{z}{c^2}\cos\alpha\cos\gamma}{\dfrac{\cos^2\alpha}{a^2} + \dfrac{\cos^2\beta}{b^2} + \dfrac{\cos^2\gamma}{c^2}}\, d\omega$$

$$= 2\rho \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \frac{\dfrac{x}{a^2}\cos^2\theta\sin\theta\, d\theta\, d\phi}{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta\cos^2\phi}{b^2} + \dfrac{\sin^2\theta\sin^2\phi}{c^2}},$$

since
$$\int_0^{2\pi} \frac{\cos\phi\, d\phi}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta\cos^2\phi}{b^2} + \frac{\sin^2\theta\sin^2\phi}{c^2}}$$
$$= \int_0^{2\pi} \frac{\sin\phi\, d\phi}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta\cos^2\phi}{b^2} + \frac{\sin^2\theta\sin^2\phi}{c^2}} = 0.$$

The components Y and Z are obtained in like manner, and the subsequent reductions are the same as those in the preceding Articles.

It is obvious that X may be expressed by the equation

$$X = 2\rho\frac{x}{a^2}\int \frac{\cos^2 a\, d\omega}{\frac{\cos^2 a}{a^2} + \frac{\cos^2\beta}{b^2} + \frac{\cos^2\gamma}{c^2}}. \qquad (18)$$

24. **Components of Attraction expressed by Elliptic Functions.**—The expressions for X, Y, Z given by equations (15) can be made to depend on elliptic functions of the first and second kind.

In order to effect this transformation, assume $\lambda_1 u = \tan\psi$, and let

$$a^2 - b^2 = h^2,\;\; a^2 - c^2 = k^2,\;\; \frac{a^2 - b^2}{a^2 - c^2} = \kappa^2,\;\; \sqrt{1 - \kappa^2\sin^2\psi} = \Delta(\psi) = \Delta.$$

then
$$u^2 = \frac{\tan^2\psi}{\lambda_1^2},\quad du = \frac{\sec^2\psi}{\lambda_1}\,d\psi,\quad 1 + \lambda_1^2 u^2 = \sec^2\psi,$$

$$1 + \lambda_2^2 u^2 = \left(1 - \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2}\sin^2\psi\right)\sec^2\psi = \sec^2\psi\,\Delta^2;$$

hence, if the equations for X, Y, Z be written in the form

$$X = \frac{3M}{c^3}xI_1,\;\; Y = \frac{3M}{c^3}yI_2,\;\; Z = \frac{3M}{c^3}zI_3, \qquad (19)$$

we have
$$\lambda_1^3 I_1 = \int\frac{\sin^2\psi\, d\psi}{\Delta(\psi)} = \frac{1}{\kappa^2}\int\frac{1 - (1 - \kappa^2\sin^2\psi)}{\Delta(\psi)}\,d\psi$$
$$= \frac{1}{\kappa^2}\left\{\int\frac{d\psi}{\Delta(\psi)} - \int\Delta(\psi)\,d\psi\right\}. \qquad (20)$$

In the ordinary notation of elliptic functions

$$\int_0^\psi \frac{d\psi}{\Delta(\psi)} = F(\psi), \text{ and } \int_0^\psi \Delta(\psi)\, d\psi = E(\psi).$$

Introducing this notation we have

$$I_1 = \frac{F(\psi_1) - E(\psi_1)}{\lambda_1(\lambda_1^2 - \lambda_2^2)}, \tag{21}$$

where $\psi_1 = \tan^{-1}\lambda_1$. The values of I_2 and I_3 can be obtained directly from (15), but it is easier to deduce them from that of I_1 in the following manner:—

From (18) and the corresponding equations for Y and Z we have

$$\frac{X}{x} + \frac{Y}{y} + \frac{Z}{z} = 2\rho \int d\omega = 4\pi\rho, \tag{22}$$

$$a^2\frac{X}{x} + b^2\frac{Y}{y} + c^2\frac{Z}{z} = 2\rho\, a^2b^2c^2 \int \frac{d\omega}{a^2b^2\cos^2\gamma + b^2c^2\cos^2\alpha + c^2a^2\cos^2\beta}.$$

From the first of these equations we get

$$I_1 + I_2 + I_3 = 4\pi\rho\frac{c^3}{3M} = \frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}, \tag{23}$$

and from the second, by means of reductions such as are applied to the expression for Z_3 in Art. 21, we have

$$\frac{3M}{c^3}\left\{(1+\lambda_1^2)I_1 + (1+\lambda_2^2)\, I_2 + I_3\right\} = \frac{4\pi\rho ab}{c^2}\int_0^1 \frac{du}{(1+\lambda_1^2u^2)^{\frac{1}{2}}(1+\lambda_2^2u^2)^{\frac{1}{2}}}$$

By a transformation similar to that applied above to the expression for I_1 this equation becomes

$$(1+\lambda_1^2)\, I_1 + (1+\lambda_2^2)\, I_2 + I_3 = \frac{F(\psi_1)}{\lambda_1}. \tag{24}$$

Hence, by (23), we have

$$\lambda_1^2 I_1 + \lambda_2^2 I_2 = \frac{F(\psi_1)}{\lambda_1} - \frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}. \tag{25}$$

The value of I_1 being given by (21), if we solve for I_2 and I_3 from (23) and (25) we get

$$\left.\begin{aligned} I_2 &= \frac{\lambda_1 E(\psi_1)}{\lambda_2^2(\lambda_1^2-\lambda_2^2)} - \frac{F(\psi_1)}{\lambda_1(\lambda_1^2-\lambda_2^2)} - \frac{1}{\lambda_2^2\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}} \\ I_3 &= \frac{1}{\lambda_2^2}\sqrt{\frac{1+\lambda_2^2}{1+\lambda_1^2}} - \frac{E(\psi_1)}{\lambda_1\lambda_2^2} \end{aligned}\right\}. \quad (26)$$

Hence, remembering that

$$\lambda_1 = \tan\psi_1, \quad c^2\lambda_1^2 = k^2, \quad c^2\lambda_2^2 = k^2 - h^2,$$

from equations (19) we obtain

$$\left.\begin{aligned} X &= \frac{3Mx}{h^2k}\{F(\psi_1) - E(\psi_1)\}, \\ Y &= 3My\left\{\frac{kE(\psi_1)}{h^2(k^2-h^2)} - \frac{F(\psi_1)}{h^2k} - \frac{\sin\psi_1\cos\psi_1}{k(k^2-h^2)\,\Delta(\psi_1)}\right\}, \\ Z &= \frac{3Mz}{k(k^2-h^2)}\{\tan\psi_1\,\Delta(\psi_1) - E(\psi_1)\} \end{aligned}\right\}. \quad (27)$$

Equations (27) assume a simpler form in the case of an *ellipsoid of revolution*. Such an ellipsoid may be either oblate or prolate.

In the case of an oblate ellipsoid of revolution $a = b$, and therefore $\lambda_1 = \lambda_2 = \lambda$. Hence $F(\psi_1) = \psi_1 = \tan^{-1}\lambda$, and equations (23) and (25) become

$$2I_1 + I_3 = \frac{1}{1+\lambda^2}, \quad 2\lambda^2 I_1 = \frac{\psi_1}{\lambda} - \frac{1}{1+\lambda^2},$$

whence we obtain

$$I_1 = \frac{1}{2\lambda^3}\left(\tan^{-1}\lambda - \frac{\lambda}{1+\lambda^2}\right), \quad I_3 = \frac{1}{\lambda^3}(\lambda - \tan^{-1}\lambda),$$

and therefore we have

$$\left.\begin{aligned}X &= \frac{3Mx}{2\lambda^3 c^3}\left\{\psi_1 - \sin\psi_1\cos\psi_1\right\} = \frac{3Mx}{2\lambda^3 c^3}\left\{\tan^{-1}\lambda - \frac{\lambda}{1+\lambda^2}\right\},\\ Y &= \frac{3My}{2\lambda^3 c^3}\left\{\psi_1 - \sin\psi_1\cos\psi_1\right\} = \frac{3My}{2\lambda^3 c^3}\left\{\tan^{-1}\lambda - \frac{\lambda}{1+\lambda^2}\right\},\\ Z &= \frac{3Mz}{\lambda^3 c^3}\left\{\tan\psi_1 - \psi_1\right\} = \frac{3Mz}{\lambda^3 c^3}\left\{\lambda - \tan^{-1}\lambda\right\}\end{aligned}\right\}. \quad (28)$$

In the case of a prolate ellipsoid of revolution $b = c$, and therefore $\lambda_2 = 0$. Putting $\lambda_1 = \lambda$, we have, then,

$$F(\psi_1) = \int_0^{\psi_1}\frac{d\psi}{\cos\psi} = \log\frac{1+\sin\psi_1}{\cos\psi_1} = \log\left\{\lambda + \sqrt{1+\lambda^2}\right\} = \chi,$$

where $\tan\psi_1 = \lambda = \sinh\chi$. Hence equations (23) and (25) become

$$I_1 + 2I_3 = \frac{1}{\sqrt{1+\lambda^2}}, \quad \lambda^2 I_1 = \frac{\chi}{\lambda} - \frac{1}{\sqrt{1+\lambda^2}},$$

whence $$I_3 = \frac{1}{2\lambda^3}\left\{\lambda\sqrt{1+\lambda^2} - \chi\right\},$$

and therefore

$$\left.\begin{aligned}X &= \frac{3Mx}{\lambda^3 c^3}\left\{\chi - \tanh\chi\right\} = \frac{3Mx}{\lambda^3 c^3}\left(\log(\lambda + \sqrt{1+\lambda^2}) - \frac{\lambda}{\sqrt{1+\lambda^2}}\right)\\ Y &= \frac{3My}{2\lambda^3 c^3}\left\{\sinh\chi\cosh\chi - \chi\right\}\\ &= \frac{3My}{2\lambda^3 c^3}\left\{\lambda\sqrt{1+\lambda^2} - \log(\lambda + \sqrt{1+\lambda^2})\right\}\\ Z &= \frac{3Mz}{2\lambda^3 c^3}\left\{\sinh\chi\cosh\chi - \chi\right\}\\ &= \frac{3Mz}{2\lambda^3 c^3}\left\{\lambda\sqrt{1+\lambda^2} - \log(\lambda + \sqrt{1+\lambda^2})\right\}\end{aligned}\right\}. \quad (29)$$

The methods employed in this Article are due to Mr. F. Purser.

Examples.

1. If X, Y, Z be the components of the attraction of a homogeneous ellipsoid at an internal point, whose coordinates are $-x$, $-y$, $-z$, show geometrically that

$$\frac{X}{x} + \frac{Y}{y} + \frac{Z}{z} = 4\pi\rho,$$

where ρ is the density of the ellipsoid.

If R_1, R_2, and R_3 be three parallel chords drawn through the extremities of the axes of an ellipsoid, and making with them the angles α, β, and γ, we have

$$\frac{R_1 \cos\alpha}{2a} + \frac{R_2 \cos\beta}{2b} + \frac{R_3 \cos\gamma}{2c} = 1,$$

whence
$$\frac{X_1}{a} + \frac{Y_2}{b} + \frac{Z_3}{c} = 4\pi\rho,$$

from which equation that given above follows by Art. 21.

2. If a straight line be drawn from the centre to the surface of a homogeneous ellipsoid, prove that the directions of the resultant attraction at all points of the line are parallel.

3. Prove that the attraction of a homœoid at a point on its external surface is in the direction of the normal, and is expressed by $4\pi\rho\delta$, where δ is the thickness of the homœoid at the point.

By Art. 18, the attraction at a point P inside and infinitely near the internal surface of the homœoid is zero. As P passes through the homœoid to the outside, the normal component of attraction is altered by $4\pi\sigma$, Art 16, the other components remaining unchanged. Hence, outside and infinitely near the external surface, the total attraction is in the direction of the normal and is equal to $4\pi\sigma$, but in this case $\sigma = \rho\delta$.

4. Show that the attraction of a homœoid at a point on its external surface varies directly as the central perpendicular on the tangent plane at the point.

If p denote the central perpendicular on a tangent plane, α, β, γ its direction angles, and a, b, c the semi-axes of the ellipsoid, we have

$$p^2 = a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2 \cos^2\gamma.$$

Hence, if a, b, c vary, α, β, γ remaining constant,

$$p\,dp = a\,da \cos^2\alpha + b\,db \cos^2\beta + c\,dc \cos^2\gamma = a^2\cos^2\alpha \frac{da}{a} + b^2 \cos^2\beta \frac{db}{b} + c^2 \cos^2\gamma \frac{dc}{c};$$

but in passing from an ellipsoid to a consecutive similar ellipsoid

$$\frac{da}{a} = \frac{db}{b} = \frac{dc}{c}, \text{ and therefore } p\,dp = p^2 \frac{da}{a}, \text{ whence } \delta = dp = p\frac{da}{a};$$

and if R denote the attraction of the homœoid at the point of contact of the tangent plane,

$$R = 4\pi\rho\delta = 4\pi\rho \frac{da}{a} p.$$

5. A mass of homogeneous fluid, subject to its own attraction, is rotating as a rigid body with a uniform angular velocity; prove that its external surface may be an oblate ellipsoid of revolution having its shortest axis coincident with the axis of rotation.

When a fluid is in equilibrium, its free surface must be perpendicular at each point to the resultant force. If the fluid be in motion, by D'Alembert's Principle (see "Dynamics," chap. ix.), the acting forces, together with the forces of inertia, form a system in equilibrium. In the present case, if x, y, z be the coordinates of a point on the free surface of the fluid, dx, dy, dz are proportional to the direction cosines of a tangent at this point; and if the axis of rotation be taken as the axis of z, and ω denote the angular velocity, the components of the acceleration of a particle of fluid at x, y, z are $-\omega^2 x$ and $-\omega^2 y$. Hence, if X, Y, Z denote the components of the force per unit of mass acting at the point x, y, z, we have

$$(X + \omega^2 x)\,dx + (Y + \omega^2 y)\,dy + Z dz = 0. \qquad (a)$$

This equation can, in general, be satisfied by supposing the free surface to be an oblate ellipsoid of revolution whose equation is

$$\frac{x^2 + y^2}{c^2(1 + \lambda^2)} + \frac{z^2}{c^2} = 1.$$

The attraction components X, Y, Z are then of the form $-Ax$, $-Ay$, and $-Cz$, where A and C are given by equations (28), and the differential equation of the ellipsoid is

$$x\,dx + y\,dy + (1 + \lambda^2)\,z\,dz = 0.$$

This is identical with the equation to be satisfied at the free surface provided that $(1 + \lambda^2)(A - \omega^2) = C$. If ρ, the density of the fluid, and ω^2 be given, this equation determines λ. Substituting for A and C from (28), remembering that

$$M = \frac{4}{3}\pi\rho c^3 (1 + \lambda^2), \quad \text{and putting} \quad \frac{\omega^2}{4\pi\rho} = q, \quad \text{we have}$$

$$\tan^{-1}\lambda = \frac{3\lambda + 2q\lambda^3}{3 + \lambda^2}.$$

For a discussion of this equation see Laplace, "Mécanique Céleste," Livre III., chap. iii. Laplace uses q to denote $\frac{3\omega^2}{4\pi\rho}$.

6. An ellipsoid with three unequal axes, and having its shortest axis as the axis of rotation, is a possible form of relative equilibrium for a revolving mass of homogeneous fluid.

The components of the attraction of an ellipsoid at a point x, y, z on its surface are of the form $-Ax$, $-By$, and $-Cz$. If we substitute these expressions for X, Y, Z in (*a*), Ex. 5, we have

$$(A-\omega^2)\,x\,dx + (B-\omega^2)\,y\,dy + Cz\,dz = 0.$$

The differential equation of the surface of the ellipsoid is

$$\frac{x\,dx}{1+\lambda_1^2} + \frac{y\,dy}{1+\lambda_2^2} + z\,dz = 0,$$

and this equation is identical with the former provided that

$$C = (1+\lambda_1^2)(A-\omega^2) = (1+\lambda_2^2)(B-\omega^2).$$

If we eliminate ω^2 from these equations, we have

$$(1+\lambda_1^2)(1+\lambda_2^2)(A-B) = (\lambda_2^2-\lambda_1^2)\,C.$$

Substituting for A, B, C from (15), and putting

$$\sqrt{(1+\lambda_1^2 u^2)(1+\lambda_2^2 u^2)} = U,$$

we get

$$(1+\lambda_1^2)(1+\lambda_2^2)\int_0^1\left(\frac{1}{1+\lambda_1^2 u^2} - \frac{1}{1+\lambda_2^2 u^2}\right)\frac{u^2\,du}{U} = (\lambda_2^2-\lambda_1^2)\int_0^1 \frac{u^2\,du}{U}.$$

Transposing and reducing, we have

$$(\lambda_1^2-\lambda_2^2)\int_0^1 \frac{(1-u^2)(1-\lambda_1^2\lambda_2^2u^2)\,u^2\,du}{U^3} = 0.$$

If λ_1 be given, one solution of this equation is $\lambda_2 = \lambda_1$, which corresponds to an ellipsoid of revolution. If this solution be rejected, and if we put

$$(\lambda_1+\lambda_2)\int_0^1 \frac{(1-u^2)(1-\lambda_1^2\lambda_2^2u^2)\,u^2\,du}{U^3} = f(\lambda_2),$$

we see that $f(\lambda_2)$ is positive when $\lambda_2 = 0$, and negative when $\lambda_2 = \infty$. Hence $f(\lambda_2)$ must vanish for some positive value of λ_2. This value of λ_2 gives a real ellipsoid satisfying the conditions of the question, provided the corresponding values of λ_1 and ω^2 are real and positive. From the equations of condition given above we find

$$\omega^2 = A - \frac{C}{1+\lambda_1^2} = \frac{\lambda_1^2}{1+\lambda_1^2}\,\frac{3M}{c^3}\int_0^1 \frac{(1-u^2)\,u^2\,du}{(1+\lambda_1^2u^2)\,U},$$

whence it appears that if λ_1 be assigned, the corresponding value of ω^2 is real and positive. The above theorem is due to Jacobi.

7. If the figure of a revolving mass of homogeneous fluid in relative equilibrium be an ellipsoid, its shortest axis must be the axis of rotation.

A particle of fluid at the extremity of the axis of rotation has no acceleration, and therefore the resultant attraction at this point is normal to the surface; but from equations (15) or (27), it is plain that the resultant attraction of an ellipsoid is not normal to the surface except at the extremity of an axis. Hence the axis of rotation is an axis of the ellipsoid. If it be not the shortest axis, let it be the axis of x; then the equation to be satisfied at the free surface is

$$Ax\,dx + (B-\omega^2)\,y\,dy + (C-\omega^2)\,z\,dz = 0,$$

which must therefore be identical with

$$\frac{x\,dx}{1+\lambda_1^2} + \frac{y\,dy}{1+\lambda_2^2} + z\,dz = 0;$$

whence $$(1+\lambda_1^2)\,A = C - \omega^2,$$

and therefore $$\omega^2 = C - (1+\lambda_1^2)\,A = -\frac{3M}{c^3}\,\lambda_1^2 \int_0^1 \frac{(1-u^2)\,u^2\,du}{(1+\lambda_1^2 u^2)\,U};$$

but this, being a negative quantity, is an impossible value for ω^2. If the axis of y were the axis of rotation, we should have, in like manner,

$$\omega^2 = -\frac{3M}{c^3}\,\lambda_2^2 \int_0^1 \frac{(1-u^2)\,u^2\,du}{(1+\lambda_2^2 u^2)\,U}.$$

Hence the only possible axis of rotation is the shortest axis of the ellipsoid.

8. An approximately spherical ellipsoid having three unequal axes is not a possible form of equilibrium for a revolving mass of homogeneous fluid.

If an ellipsoid having unequal axes be approximately spherical, λ_1 and λ_2 must be both small quantities, less than unity; but $f(\lambda_2)$ in Ex. 6 is positive when $\lambda_2 = 0$, and remains positive so long as $\lambda_1^2\lambda_2^2 < 1$. Hence the value of λ_2 for which $f(\lambda_2)$ vanishes is greater than $\frac{1}{\lambda_1^2}$; and therefore if λ_1 be less than unity, λ_2 must be greater than unity for the ellipsoid which is the figure of equilibrium. This ellipsoid cannot therefore be approximately spherical.

The theorem above is proved by Laplace by the use of spherical harmonics.

9. A prolate ellipsoid of revolution is not a possible form of relative equilibrium for a revolving mass of homogeneous fluid.

If the axis of revolution of the ellipsoid be the axis of rotation, the theorem is a particular case of Ex. 7, and is given by Laplace, "Mécanique Céleste," Livre III., chap. iii.

If the axis of rotation be perpendicular to the axis of revolution of the ellipsoid, $f(\lambda_2)$ in Ex. 6 should vanish when $\lambda_2 = 0$; but this is not the case, and therefore under no circumstances can a prolate ellipsoid of revolution be a figure of equilibrium.

10. Prove that the direction of the resultant attraction at any point of the surface of a prolate ellipsoid of revolution lies between the normal to the surface at the point and the line drawn from it to the centre of the ellipsoid.

Let α be the angle which the resultant attraction at any point of the surface of a homogeneous prolate ellipsoid of revolution makes with the axis of revolu-

tion which is the axis of x. Then x and y being the coordinates of this point in the plane of the generating ellipse passing through it, we have, by (29),

$$\tan\alpha = \frac{\sinh\chi\cosh\chi - \chi}{2(\chi - \tanh\chi)}\,\frac{y}{x};$$

also if ν be the angle which the normal at the same point makes with the axis of x, we have

$$\tan\nu = (1+\lambda^2)\frac{y}{x} = \frac{y}{x}\cosh^2\chi.$$

If we compare the expressions for $\tan\alpha$ and $\tan\nu$, we see that $\nu > \alpha$, provided $F(\chi)$ is positive, where

$$F(\chi) = \chi(2\cosh^2\chi + 1) - 3\sinh\chi\cosh\chi.$$

Now $$F(\chi) = \chi(2 + \cosh 2\chi) - \tfrac{3}{2}\sinh 2\chi,$$

and if we substitute for $\cosh 2\chi$ and $\sinh 2\chi$ their expansions in powers of 2χ, we see that $F(\chi)$ is positive when χ is positive, and χ must be positive, since it is the logarithm of a quantity greater than unity.

Again, $\tan\alpha > \dfrac{y}{x}$, provided that $\sinh\chi\cosh\chi - \chi > 2(\chi - \tanh\chi)$.

This condition is fulfilled if $f(\chi)$ be positive, where

$$f(\chi) = \sinh\chi\cosh^2\chi + 2\sinh\chi - 3\chi\cosh\chi;$$

but here $$f(\chi) = \tfrac{1}{4}(\sinh 3\chi + 9\sinh\chi) - 3\chi\cosh\chi;$$

and expanding in powers of χ we see, as before, that $f(\chi)$ is positive.

By a similar procedure the same result can be obtained in the case of an oblate ellipsoid of revolution.

11. Find the mutual attraction between the two hemispheres into which a homogeneous sphere is divisible by a plane through its centre.

If the sphere were fluid, it would be in equilibrium under its own attraction; then the fluid pressure p at any point would be given by the equation

$$p = \rho\int(Xdx + Ydy + Zdz) + C = -\rho\int\frac{4\pi\rho}{3}\,r\,dr + C = C - \tfrac{2}{3}\pi\rho^2 r^2;$$

but $p = 0$ when $r = a$ the radius of the sphere;

$$\therefore\quad p = \tfrac{2}{3}\pi\rho^2(a^2 - r^2).$$

Again in this case the entire pressure P perpendicular to a central section of the sphere is given by the equation

$$P = \tfrac{2}{3}\pi\rho^2\int_0^a\int_0^{2\pi}(a^2 - r^2)\,r\,dr\,d\theta = \frac{\pi^2\rho^2 a^4}{3}.$$

Now suppose one hemisphere of the fluid sphere rigidified, it will remain in equilibrium under the same external forces as those which acted on it before;

and therefore the mutual attraction F between the hemispheres must be equal to P. Hence, if M be the mass of one hemisphere, we have

$$F = \tfrac{3}{4}\frac{M^2}{a^2}.$$

Another method of arriving at this result will be found in Ex. 14.

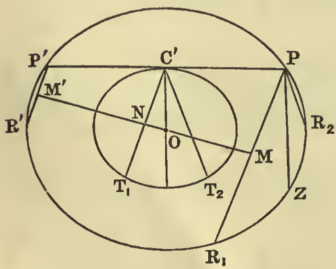
12. Prove geometrically that the component, parallel to an axis, of the attraction of an ellipsoid E at a point P on its surface, is equal to the attraction of an ellipsoid E', similar to E and similarly placed, at a point at the extremity of the parallel axis, provided this axis be equal to the parallel chord of E drawn through P.

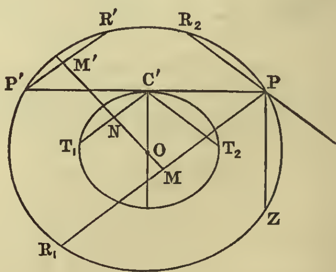
Let PZ be the chord of E parallel to an axis of E, and let C' be the extremity of the codirectional axis of E', then, O being the centre of the ellipsoids E and E', we have $2OC' = PZ$.

Draw any plane through PZ, it meets the ellipsoid E in an ellipse whose axis is parallel to PZ, and the parallel plane through OC' meets the ellipsoid E' in a similar ellipse. Suppose these two ellipses put in one plane and made concentric and similarly placed, then the axis of the inner is equal to the parallel chord PZ of the outer.

Draw two chords PR_1 and PR_2 making equal angles with PZ on opposite sides, join PC', and from the point P' where it meets the outer ellipse again draw $P'R'$ parallel to PR_1; draw also $C'T_1$ and $C'T_2$, chords of the inner ellipse, parallel to PR_1 and PR_2. Then $P'R' = PR_2$, and the middle points of $P'R'$, $C'T'$, and PR_1 are in one straight line, and C' is the middle point of PP', wherefore

$$C'T_1 + C'T_2 = 2C'T_1 = PR_1 + P'R' = PR_1 + PR_2$$



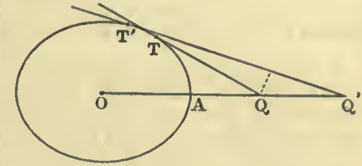


when R_1 and R_2 are on opposite sides of PZ; but

$$2C'T_1 = PR_1 - P'R' = PR_1 - PR_2$$

when R_1 and R_2 are on the same side of PZ. Hence, as the attraction of a thin cone at its vertex is proportional to its length, the truth of the theorem is manifest.

13. Show how to represent the attraction of a homogeneous ellipsoid at a point P at the extremity of its axis by means of an arc of one of its focal conics and the tangent drawn at the extremity of this arc.



If P be at the end of the shortest axis $2c$ of the ellipsoid, assume a point Q on the longest axis such that, O being the centre,

$$OQ = c\lambda_1 \sqrt{1 + \lambda_2^2 u^2},$$

and draw through Q a tangent QT to the focal ellipse; then, if ψ be the angle which the perpendicular p from O on QT makes with the axis-major of the ellipse, putting $OQ = \xi$, we have

$$\xi^2 \cos^2\psi = p^2 = c^2 (\lambda_1^2 \cos^2\psi + \lambda_2^2 \sin^2\psi).$$

Substituting for ξ we have

$$\lambda_1^2 \lambda_2^2 u^2 = (\lambda_1^2 \lambda_2^2 u^2 + \lambda_2^2) \sin^2\psi;$$

whence

$$\sin\psi = \frac{\lambda_1 u}{\sqrt{(1 + \lambda_1^2 u^2)}}, \quad \text{also } d\xi = \frac{c\lambda_1 \lambda_2^2 u\, du}{\sqrt{(1 + \lambda_2^2 u^2)}}.$$

If A be the extremity of the axis major of the focal ellipse, and if $QT = t$, arc $AT = s$, it is plain, by drawing a consecutive tangent $Q'T'$ whose length is t', that $t' - t = \sin\psi d\xi + s' - s$, whence $\sin\psi d\xi = dt - ds$. Now if Z_3 be the attraction of the ellipsoid at P, by (14), Art. 21, we have

$$Z_3 = \frac{3M}{c^3 \lambda_1^2 \lambda_2^2} \cdot \int \sin\psi d\xi = \frac{3Mc}{(a^2 - c^2)(b^2 - c^2)} (t - s),$$

where t is drawn from a point K on the axis such that

$$OK = \frac{\sqrt{a^2 - c^2}}{c} b.$$

If the point P be at the extremity of the mean axis, assume Q on the greatest axis so that

$$OQ = \sqrt{a^2 - b^2} \sqrt{\left(1 + \frac{c^2 - b^2}{b^2} u^2\right)};$$

then, proceeding in a manner similar to that above, we find

$$Y_2 = \frac{3Mb}{(a^2 - b^2)(b^2 - c^2)} (t - s),$$

where t is the tangent drawn from a point K such that

$$OK = \frac{\sqrt{a^2 - b^2}}{b} c,$$

and s is the corresponding arc of the focal hyperbola.

The attraction at the extremity of the longest axis can be found from those at the extremities of the other two by means of the relation given in Ex. 1. The construction in this example is due to Mac Cullagh.

14. Find the mutual attraction between the portions into which a homogeneous ellipsoid is divided by a central plane perpendicular to a principal axis.

Let E_1 and E_2 be the semi-ellipsoids into which the ellipsoid E is divided by the diametral plane. The attraction of E on E_2 is compounded of the attractions of E_1 on E_2, and of E_2 on itself, but the latter is zero, being the resultant of pairs of equal and opposite forces, and therefore the attraction R of E_1 on E_2 is equal to that of the whole ellipsoid on E_2.

If the diametral plane be perpendicular to the shortest axis of the ellipsoid, then, for the corresponding attraction R_3, we have $R_3 = C\rho \iiint z\,dx\,dy\,dz$ taken through the volume of the semi-ellipsoid, C being given by equations (15). If we change the variables by assuming

$$\frac{x}{a} = \frac{\xi}{k}, \quad \frac{y}{b} = \frac{\eta}{k}, \quad \frac{z}{c} = \frac{\zeta}{k},$$

we obtain
$$R_3 = C\rho\,\frac{abc^2}{k^4}\int \zeta\,d\xi\,d\eta\,d\zeta$$

taken through the volume of the hemisphere whose radius is k; whence

$$R_3 = \frac{\pi\rho abc^2}{4}\,C = \frac{3M}{16}\,Cc,$$

where M denotes the mass of the ellipsoid, and C is given by (15) or (27).

In like manner, for diametral planes perpendicular to the other two axes of the ellipsoid, we have

$$R_1 = \frac{3M}{16}\,Aa, \quad R_2 = \frac{3M}{16}\,Bb.$$

15. Show that the mutual attraction of the two semi-ellipsoids situated on opposite sides of a principal plane is equal to the attraction of the entire ellipsoid on a particle of half its own mass situated at the centre of inertia of one of the semi-ellipsoids.

16. Show how to determine the attraction of a homogeneous solid of revolution at any point O on its axis.

If we take the point O for origin, the axis of revolution for axis of x, and a perpendicular to it for that of y, and if X denote the required attraction, by (3), Art. 14, we have

$$X = 2\pi\rho\int\left\{1 - \frac{x}{\sqrt{(x^2+y^2)}}\right\}dx.$$

17. Determine the form of the homogeneous solid of revolution, of given density and mass, whose attraction at a point O on the axis of revolution is the greatest possible.

The axes being taken as in the last example, the mass of the solid is expressed by $\pi\rho\int y^2\,dx$. Hence

$$\int\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)dx+\alpha\int y^2\,dx$$

is to be made a maximum, where α is constant.

Taking the variation of the quantity under the integral sign, and equating to zero the coefficient in it of δy, we get

$$\alpha y+\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}=0,$$

whence, putting $\alpha=-\frac{1}{a^2}$, we obtain for the equation of the curve which generates the required solid

$$a^2x=r^3, \quad \text{or} \quad \frac{\cos\theta}{r^2}=\frac{1}{a^2}.$$

This curve passes through the origin O and cuts the axis of x there at right angles.

A curve whose equation is $\frac{\cos\theta}{r^2}=C$, where C is constant, obviously possesses the property that an element of mass, wherever it is placed on this curve, produces at the origin the same component of attraction parallel to the axis of x. By varying θ and keeping r constant, we see that, as C increases, the corresponding curve approaches the axis. Hence, if an element of mass be moved from one of these curves to another lying inside the former, the attraction component due to this element is increased. It is now easy to verify the result obtained already by the Calculus of Variations; for if the generating curve A of the surface bounding a solid M having the required mass be not a curve of the above form, describe a curve B of this form, such that the corresponding solid has the mass required; then, if the mass in that part of A lying outside B be moved into the space inside B not enclosed by A, the attraction component of M is increased, and is therefore a maximum when A coincides with B.

18. An uniplanar mass m, acting inversely as the distance, is placed at a point on the circumference of a circle whose centre is O, and which passes through a point P; show that the force which m exerts at P in the direction PO is the same wherever m be situated.

19. A given amount of uniplanar mass is homogeneously distributed so as to produce the greatest possible attraction at at a given point P; the density of the distribution being assigned, find the form of the curve by which it is bounded. *Ans.* A circle passing through P.

20. Deduce equation (25), Art. 24, by direct integration from the values of I_1 and I_2 given by (15), Art. 21.

By (15) we have

$$\lambda_1^2 I_1=\int_0^1\frac{\lambda_1^2u^2\,du}{(1+\lambda_1^2u^2)^{\frac{3}{2}}\,(1+\lambda_2^2u^2)^{\frac{1}{2}}}$$

$$=-\int\frac{u}{(1+\lambda_2^2u^2)^{\frac{1}{2}}}\,d\,(1+\lambda_1^2u^2)^{-\frac{1}{2}}$$

$$= \frac{-1}{(1+\lambda_1^2)^{\frac{1}{2}}(1+\lambda_2^2)^{\frac{1}{2}}} - \int_0^1 \frac{\lambda_2^2 u^2\, du}{(1+\lambda_1^2 u^2)^{\frac{1}{2}}(1+\lambda_2^2 u^2)^{\frac{3}{2}}} + \int_0^1 \frac{du}{(1+\lambda_1^2 u^2)^{\frac{1}{2}}(1+\lambda_2^2 u^2)^{\frac{1}{2}}}$$

$$= \frac{-1}{(1+\lambda_1^2)^{\frac{1}{2}}(1+\lambda_2^2)^{\frac{1}{2}}} - \lambda_2^2 I_2 + \frac{F(\psi_1)}{\lambda_1},$$

whence (25) follows by transposition.

21. Find from equations (15) by direct integration the components of the attraction of an oblate ellipsoid of revolution at a point on its surface.

Here
$$X = \frac{3M}{c^3} x \int_0^1 \frac{u^2\, du}{(1+\lambda^2 u^2)^2}.$$

Putting $\lambda u = \tan \psi$, we have

$$X = \frac{3Mx}{\lambda^3 c^3} \int \frac{\tan^2\psi \sec^2\psi\, d\psi}{\sec^4\psi} = \frac{3Mx}{\lambda^3 c^3} \int (d\psi - \cos^2\psi\, d\psi) = \frac{3Mx}{2\lambda^3 c^3}(\psi_1 - \sin\psi_1 \cos\psi_1),$$

where $\tan\psi_1 = \lambda$. Again,

$$Z = \frac{3Mz}{c^3} \int_0^1 \frac{u^2\, du}{1+\lambda^2 u^2} = \frac{3Mz}{\lambda^3 c^3} \int \tan^2\psi\, d\psi = \frac{3Mz}{\lambda^3 c^3}(\tan\psi_1 - \psi_1).$$

22. Find from equations (15) by direct integration the components of the attraction of a prolate ellipsoid of revolution at a point on its surface.

In this case
$$X = \frac{3Mx}{c^3} \int_0^1 \frac{u^2\, du}{(1+\lambda^2 u^2)^{\frac{3}{2}}},$$

$$Y = \frac{3My}{c^3} \int_0^1 \frac{u^2\, du}{(1+\lambda^2 u^2)^{\frac{1}{2}}}.$$

Putting $\lambda u = \sinh \chi$, we have

$$X = \frac{3Mx}{\lambda^3 c^3} \int \frac{\sinh^2\chi \cosh\chi\, d\chi}{\cosh^3\chi}$$

$$= \frac{3Mx}{\lambda^3 c^3} \int (d\chi - \operatorname{sech}^2\chi\, d\chi) = \frac{3Mx}{\lambda^3 c^3}(\chi_1 - \tanh\chi_1),$$

$$Y = \frac{3My}{\lambda^3 c^3} \int \frac{\sinh^2\chi \cosh\chi\, d\chi}{\cosh\chi}$$

$$= \frac{3My}{2\lambda^3 c^3} \int (\cosh 2\chi\, d\chi - d\chi) = \frac{3My}{2\lambda^3 c^3}(\sinh\chi_1 \cosh\chi_1 - \chi_1),$$

where $\sinh\chi_1 = \lambda$.

CHAPTER III.

LINES OF FORCE.

25. **Field of Force.**—A region of space considered in reference to the action of attracting or repelling masses is called a *field of force.*

If a curve be drawn in a field such that the tangent at each point is co-directional with the resultant force at that point, the curve is called a *line of force.*

If x, y, z denote the coordinates of a point in the field, and X, Y, Z the components of the resultant force at that point, the differential equations of a line of force are plainly

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}. \tag{1}$$

So long as a line of force does not pass through mass it is continuous. In the case of a volume distribution, the resultant force is always continuous both in magnitude and direction; but X, Y, Z are *not*, in general, the same functions of the coordinates inside and outside the acting mass. We have seen (Art. 13) that this is so in the case of a homogeneous sphere. The equations of a line of force in general, therefore, become different when the line passes from space occupied by mass into that which is unoccupied, but the two curves have a *common tangent* at the boundary of the mass.

In the case of a sphere, it happens that the lines of force both inside and outside the sphere are straight, and so in this case the one curve is the continuation of the other.

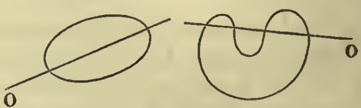
When a field of force is due partly or entirely to a surface distribution, the resultant force is discontinuous at the surface, and therefore in passing through the surface the line of force either stops altogether or *changes its direction abruptly.*

A line of force at no point of which the resultant force vanishes, cannot be a closed curve, if the force be due to permanent attracting or repelling masses.

For if it were, an element of mass placed on the curve would be driven round and round continually, and so an unlimited amount of work could be produced without the consumption of materials or using up of energy, which is impossible.

26. **Gauss' Theorem.**—For any system of mass, attracting or repelling with a force varying inversely as the square of the distance, if N be the component of the resultant force along the outward drawn normal to a closed surface S, the integral $\int NdS$ taken over the entire surface is equal to $4\pi M + 2\pi M'$, where M is the sum of the masses inside, and M' the sum of those on the surface, and those masses which *attract* are regarded as negative.

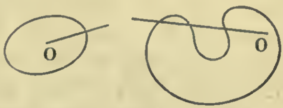
To prove this, let us consider the force due to the mass m concentrated at any point O. If O be outside S, and if r be any radius vector drawn from O which penetrates inside the surface, it must pass out again, and corresponding to every entrance there is an exit.



If we now suppose a cone of infinitely small solid angle $d\omega$ to have O for its vertex and r for its axis, the element of surface which it intercepts is $\frac{r^2\,d\omega}{\cos\psi}$, where ψ is the acute angle between r and the normal to the surface element dS. Since N is the component along the normal drawn outwards, and the force exerted by m is supposed repulsive, N_1 the normal component of the force due to m at an entrance is $-\frac{m}{r_1^2}\cos\psi_1$, whence $N_1 dS_1 = -m d\omega$; again N_2 the normal component at an exit is $\frac{m}{r_2^2}\cos\psi_2$, and therefore $N_2 dS_2 = m d\omega$, whence $N_1 dS_1 + N_2 dS_2 = 0$. As the number of entrances is equal to the number of exits, the total contribution to the surface integral $\int NdS$ due to the cone having the line r for its axis is zero. A similar result holds good for any other radius vector from O, and also

for every element of mass outside S, and therefore the part of the integral $\int NdS$ due to mass outside S is zero.

If O be inside S, any radius vector r from it must pass once out of the surface S without a corresponding entrance, and if r enters S there must for each entrance be a corresponding exit; hence the whole contribution to the surface integral $\int NdS$ due to the cone having r for axis is $md\omega$. In order to get the whole portion of the surface integral due to m, we must integrate $d\omega$ all round O. In this manner we obtain $4\pi m$. A similar result holds good for every other element of mass inside S. If O be on the surface S, all radii vectores from it on one side of the tangent plane pass out of S once without any corresponding entrance. Any radius vector from O on the other side of the tangent plane either does not meet S at all or else accomplishes as many exits as entrances. Hence in this case the entire contribution of the mass m at O to the surface integral is $\int m\,d\omega$ taken over a hemisphere, that is $2\pi m$.

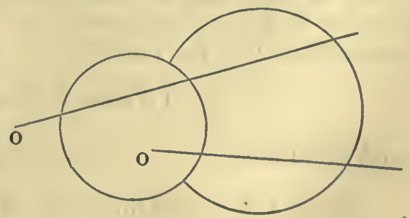


Finally, if we add together all the parts of the integral due to the various elements of mass, we obtain

$$\int NdS = 4\pi M + 2\pi M'. \qquad (2)$$

The number of entrances and exits of any one radius vector which meets the surface S depends on the form of this surface.

If the closed surface S contain two adjacent regions separated by a single sheet, this sheet must be counted twice over, once as a boundary to each region, or, which comes to the same thing, not counted at all. As an example of a surface such as has been described, we may take a sphere and the portion of another sphere terminated by the curve of intersection of the two. In this case as r passes



out of the complete sphere it may enter the region enclosed between the two spheres. The part of the complete sphere which forms the boundary between the two regions is then to be counted twice over in the estimation of the integral, and in passing through this surface r accomplishes both an exit and an entrance.

27. **Equi-potential Surfaces.**—When the acting mass is all within a finite distance of the origin, its *potential* at any point P may be defined as *the work done against the forces of the system in bringing a unit of mass* μ *from an infinite distance to the point* P, the position of the masses to which the field of force is due being supposed to be invariable.

Since in any displacement of μ perpendicular to a line of force no work is done, the potential must be the same at all points of a surface cutting the lines of force orthogonally. Hence such a surface is called an *equi-potential surface.*

28. **Tubes of Force, Solenoids.**—If through every point of a closed curve on an equipotential surface a line of force be drawn, and these lines produced to meet another equipotential surface, we obtain a *tube of force.*

In an infinitely small tube of force having no mass inside it the product of the resultant force and orthogonal section of the tube is constant.

For let Σ_1 and Σ_2 be orthogonal sections of the tube, and R_1 and R_2 the values of the resultant force at these sections. The portion of the tube intercepted between the sections Σ_1 and Σ_2 forms with them a closed surface for which $\int NdS = 0$,

but at each point of the surface made up of the lines of force $N = 0$, and at the termination Σ_1 which is perpendicular to the lines of force $N = -R_1$, whilst at Σ_2 the normal component $N = R_2$, whence

$$\int NdS = R_2\Sigma_2 - R_1\Sigma_1,$$

and therefore $$R_1\Sigma_1 = R_2\Sigma_2. \qquad (3)$$

When the geometrical form of a tube of force is known, equation (3) enables us to determine the intensity of the

resultant force at any assigned point of a line of force when the intensity at one given point is known.

If a tube of force pass through mass the product of the resultant force and orthogonal section is increased by $4\pi M$, where M is the total amount of mass which the tube contains.

For in this case

$$R_2\Sigma_2 - R_1\Sigma_1 = \int NdS = 4\pi M,$$

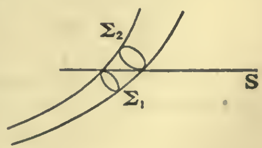
whence
$$R_2\Sigma_2 = R_1\Sigma_1 + 4\pi M. \qquad (4)$$

When a vector quantity, that is, a quantity which has direction as well as magnitude, varies throughout a certain region of space in such a way that $\int NdS$ taken over any closed surface S drawn in the region is zero, N being the component of the vector normal to the surface, this vector quantity is said to have a *solenoidal distribution.*

From what is said above it appears that the resultant force due to attractive or repulsive mass has, in unoccupied space, such a distribution. Tubes of force are accordingly sometimes called *solenoids.*

29. **Discontinuous Change of Resultant Force at Surface on which there is Finite Mass.**—We have seen already, Art. 16, that when a point P passes through a surface S on which there is a distribution of finite mass, the force component normal to the surface at P changes discontinuously by the amount $4\pi\sigma$, where σ is the surface density. The truth of this theorem, which is one of great importance, follows readily from the properties of tubes of force.

For, draw a tube of force containing the element dS, let ψ_1 and ψ_2 be the angles which its directions on each side of dS make with the normal to the element, Σ_1 and Σ_2 its corresponding orthogonal sections infinitely near dS, and Ω_1 and Ω_2 the volumes of the portions of the tube included between dS and Σ_1 and Σ_2, respectively. Then, by equation (4),



$$R_2\Sigma_2 = R_1\Sigma_1 + 4\pi\{\rho_1\Omega_1 + \rho_2\Omega_2 + \sigma dS\},$$

where ρ_1 and ρ_2 are the densities of the volume distributions

close to dS. Now $\Sigma_1 = dS \cos\psi_1$, $\Sigma_2 = dS \cos\psi_2$, and Ω_1 and Ω_2 are infinitely small quantities of an order higher than dS, and therefore

$$R_2 \cos\psi_2 = R_1 \cos\psi_1 + 4\pi\sigma. \qquad (5)$$

Hence, when P passes perpendicularly through the surface, the normal component of the resultant force in the direction in which P moves is increased by $4\pi\sigma$.

If we imagine a small closed circuit consisting of elements of lines on opposite sides of S parallel to and infinitely near a tangent, and of two consecutive normals, and if we suppose an element of mass to move round this circuit, no work is done in this displacement by the forces of the system since the circuit is closed. But the work done in one normal displacement is equal and opposite to that done in the other, hence the work done in one tangential displacement is equal and opposite to that done in the other, and therefore a tangential component of the resultant force cannot change by any finite amount in passing from one side of the surface to the other.

We can now write down the equations which hold good, at any point P of a surface on which there is a distribution of mass, between X_1, Y_1, Z_1, the components parallel to the coordinate axes of the resultant force on one side of the surface, and X_2, Y_2, Z_2, the components on the other side. For, let l, m, n denote the direction cosines of the normal, and λ, μ, ν those of any tangent to the surface at the point P, then, by what has been proved above, we have

$$l(X_2 - X_1) + m(Y_2 - Y_1) + n(Z_2 - Z_1) = 4\pi\sigma, \qquad (6)$$

$$\lambda(X_2 - X_1) + \mu(Y_2 - Y_1) + \nu(Z_2 - Z_1) = 0. \qquad (7)$$

30. **Electric Conductors and Non-Conductors.**—In reference to electric phenomena, bodies are usually divided into conductors and non-conductors. The observed properties of the former, so far as we are concerned with them, may be explained by two hypotheses:—1. All bodies contain indefinite but equal amounts of the two kinds of electric mass which an electric force tends to separate and drive in opposite directions. 2. In a conductor, electric mass moves freely in

any direction in which it is urged by the electric forces of the field, but if the conductor be insulated, that is, surrounded by non-conductors, no electric mass can leave the conductor.

As a consequence of these hypotheses, we see that there can be no force at any point in the substance of a conductor in electric equilibrium, for if there were such a force at any point P, the two kinds of electricity at P would be separated and would move in opposite directions.

Again, at a bounding surface of a conductor in electric equilibrium, the resultant force must be normal to the surface, for otherwise there would be motion of electric mass along the surface.

Hence, when an insulated conductor is brought into a field of electric force, a separation of electric mass takes place in its substance, and this separated mass is distributed in such a way as to modify the field of force so that at each point in the substance of the conductor the resultant force is zero, and that at each point on the bounding surface the resultant force is normal to the surface.

From this last result it follows that *the surface of a conductor in electric equilibrium is an equi-potential surface of the then existing electric field.*

Also, if σ be the density of the surface distribution at any point, since the resultant force inside the surface of the conductor is zero, by Art. 29 the resultant force infinitely near the surface outside is expressed by $4\pi\sigma$.

The action of electrified bodies in producing a separation and redistribution of electric mass in other bodies from which they are separated by a non-conducting medium is called *electric induction.* Faraday discovered that this action is dependent on the nature of the intervening medium.

In the case of homogeneous isotropic media and for the standard medium, in which the unit force is that between unit masses at the unit distance, results obtained from the theory of direct action at a distance according to the principles given above hold good experimentally. For other homogeneous isotropic media, the force between unit masses at the unit distance is not the unit force, but is for each medium a fixed constant, whose reciprocal is called the *specific inductive capacity of the medium.* To obtain results

experimentally correct we have then to multiply each force, when expressed as a function of masses and distances, by the constant belonging to the medium.

The true theory for *anisotropic* media, that is, media having different properties in different directions, is complicated and not fully known. See Clerk Maxwell, "Electricity and Magnetism," Part I., Chap. iv., Article 101 *a*.

In the present treatise when we have to deal with electric induction we shall suppose, except the contrary be expressly stated, that the non-conducting medium is the standard isotropic homogeneous medium.

It is easy to show by Gauss' theorem, Art. 26, that *there can be no free electric mass at any point of a conductor in electric equilibrium except the point be situated on one of its bounding surfaces.*

For, let P be a point in the conductor, describe in its substance a closed surface S_1 surrounding P, then $\int N dS_1 = 0$, since N is zero at each point of S_1; also there cannot be a distribution having a finite surface density on S_1 itself because N is zero both inside and outside S_1; therefore the total mass inside S_1 is zero, and as S_1 may be made as small as we please, it follows that there is no mass at P.

31. **Tubes of Induction.**—A tube of force is positive when its direction coincides with that of the resultant force of the field acting on positive mass.

When the field of force is due to the presence of a number of charged conductors in electric equilibrium, there is no force in the substance of any one A of these conductors; but if there be a positive charge on any element dS of its surface, a positive tube of force T, in which $\int R d\Sigma = 4\pi\sigma dS$, starts from this element.

It is possible for the resultant force to vanish at a point or line in unoccupied space. Such a point or line is a point or line of equilibrium.

It may happen that some of the lines of force bounding the tube T pass through points of equilibrium; but since $\int R d\Sigma$ is constant, there must be a continuous portion of this tube for which R does not vanish. Similar reasoning applies to this portion in its subsequent course, and we may conclude that T cannot terminate in unoccupied space, and that there

must be some lines of force inside the tube T for which R does not change its sign so long as T does not pass through mass.

We can now see that T cannot return to the conductor A; for if it did, a line of force L inside T, on which R does not change its sign, would meet the conductor A in points Q and Q'; and the circuit partly composed of L would be completed by the line $Q'Q$ on the conductor, in a displacement along which no work would be done on an element of mass by the forces of the system. Hence the forces of the system would do work in the displacement of an element of mass round a complete circuit which is impossible.

From Art. 25 it appears that a tube of force cannot be a closed tube returning into itself. On the whole, therefore, we may conclude that a positive tube of force starts from a conductor on which the charge on the portion of surface enclosed by the tube is positive, and either goes on to infinity or ends on a conductor on which the charge on the corresponding portion of surface is negative, the two charges being equal in magnitude.

The leading property of tubes of force in reference to electrical science is the numerical equality of the charges on the portions of the conductors by which they are terminated.

Tubes or solenoids along which this equality is propagated through an insulating medium are termed *tubes of induction.*

It must be remembered that the identification of tubes of force with tubes of induction in the manner effected above holds good only under the limitations laid down in the preceding Article.

In reference to electric phenomena it would seem that the existence of tubes of induction is the primary fact. This was recognized by Clerk Maxwell who indicated a method of basing on them the explanation of the observed phenomena of electricity.

Another mode of doing this has been developed by Professor J. J. Thomson. These investigations do not come within the scope of the present treatise.

In accordance with the theory adopted in this Article, *the unit tube of induction* may be defined as that in which $\int R d\Sigma$ taken over the portion of an equipotential surface enclosed by the tube is unity. The charge E on a conductor may be

indicated by the number of positive unit tubes of induction starting from its surface, that is, by the excess of the number of positive above that of negative tubes. If n be this number, we have $n = 4\pi E$. If the unit tube were defined as that which encloses on the surface of the conductor from which it starts a portion having the unit charge, we should have $n = E$. Clerk Maxwell sometimes uses one of the definitions given above, and sometimes the other ("Electricity and Magnetism," Part I., Arts. 22, 82, 89 *c*). The latter definition is that which is adopted by Professor J. J. Thomson. The tubes which have been called above tubes of induction are by him termed Faraday tubes.

In the case of tubes of *magnetic* induction, the definition of the unit tube adopted by Professor J. J. Thomson as well as by others is the first of the definitions given above.

32. **Induction over a Surface.**—If ϵ be the angle between the normal drawn outwards at any point of a surface S and the resultant force R at that point, the integral $\int R \cos \epsilon \, dS$ taken over S is called *the induction over the surface.* It is plain that if N be the force component normal to the surface, $R \cos \epsilon = N$; and we conclude, from Art. 26, that in unoccupied space the induction over a closed surface, having no mass in its interior, is zero.

Again, the induction I over a surface S bounded by a closed curve s *depends only on the curve.*

To prove this, suppose any other surface S' bounded by s, then the induction over the closed surface composed of S and S' is zero; that is, when the sides of S and S' next s are both regarded as interior, we have $I = I'$.

If the field of force result from the joint action of several force systems, the induction I over any surface S is the sum of the inductions I_1, I_2, &c., due to these systems taken separately. This is obvious since $N = N_1 + N_2 +$ &c.

If there be any two surfaces in unoccupied space, the intervening region being devoid of mass, and if the portions of these surfaces enclosed by the same tube of force, finite or otherwise, be denoted by S_1 and S_2, the induction over S_1 is equal to that over S_2. Hence we conclude that if two closed curves s_1 and s_2 lie on the same line-of-force surface the value of the induction for the one is the same as for the other.

33. **Field of Force Symmetrical round an Axis.**—When all the forces lie in planes passing through a common axis OZ, and in any two of these planes are equal respectively, and disposed in the same manner, the whole system is one of revolution round OZ.

In this case, points on the same line of force in their revolution trace out circles for which the inductions are equal.

Conversely if there be two points Q_1 and Q_2 lying in the same plane through OZ, and if the induction for the circle traced by the revolution of Q_1 be equal to that for the circle traced by Q_2, then Q_1 and Q_2 must lie on the same line of force. For if not, draw in the plane OQ_1Q_2 the equipotential curve passing through Q_2, and let it meet the line of force through Q_1 in Q'_2, then the induction over the portion of the equipotential surface lying between the circles traced by Q_2 and Q'_2 must be zero, which is impossible except there be a point of equilibrium between Q_2 and Q'_2; therefore, in general, Q_2 and Q'_2 are on the same line of force.

34. **Graphic Representation of Lines of Force.**—If the field of force be such as would be produced by a finite number of masses situated on the axis of revolution, it is always possible to obtain any number of points for which the value of the corresponding induction, as explained in Art. 33, is known.

Let A be a centre of force on the axis of revolution OZ at which the mass is m', then the lines of force for m', considered alone, are straight, and the tubes of force are cones having A as vertex. Draw through A in a plane containing OZ, a straight line AP_1 making with AZ an angle θ'_1 such that $2\pi m'(1 - \cos\theta'_1) = i$, and draw AP_2, AP_3, &c., making angles θ'_2, θ'_3, &c., with AZ such that

$$1 - \cos\theta'_1 = \cos\theta'_1 - \cos\theta'_2 = \cos\theta'_2 - \cos\theta'_3 = \&c.;$$

then since

$$\int_0^{\theta'_1}\int_0^{2\pi}\frac{m'}{a^2}a^2\sin\theta' d\theta' d\phi = 2\pi m'(1 - \cos\theta'_1),$$

and
$$\int_{\theta'_1}^{\theta'_2}\int_0^{2\pi}\sin\theta' d\theta' d\phi = 2\pi(\cos\theta'_1 - \cos\theta'_2),$$

the induction is i for each cone of revolution whose inner

and outer boundaries are generated by successive lines drawn as above. Hence, if I'_1, I'_2, &c. be the inductions due to m' for circles traced by the revolution of points on AP_1, AP_2, &c., we have

$$I'_1 = i, \quad I'_2 = 2i, \; \ldots . \; I'_{n'} = n'i.$$

In like manner for a mass m'' situated at a point B on the axis we have

$$I''_1 = i, \quad I''_2 = 2i, \; \ldots . \; I''_{n''} = n''i,$$

where I''_1, &c. correspond to points on lines BQ_1, &c. drawn through B such that

$$2\pi m''(1 - \cos\theta''_1) = i, \quad 1 - \cos\theta''_1 = \cos\theta''_1 - \cos\theta''_2 = \&\text{c}.$$

If now the field of force be due to the joint action of m' and m'', and we take the point R in which the straight line $AP_{n'}$ intersects the straight line $BQ_{n''}$, for the induction I corresponding to R, we have

$$I = I'_{n'} + I''_{n''} = (n' + n'')\, i;$$

and if the numbers n' and n'' vary, but so that $n' + n'' =$ constant $= n$, we obtain a set of points for which the corresponding induction is ni. A curve passing through these points is a line of force for the joint action of the two masses m' and m''. The points of intersection of these lines of force with the straight lines bounding the tubes of equal induction for a third mass, determine in a similar manner points for which the corresponding inductions are equal in the case of the joint action of three masses. Thus a line of force can be drawn for this case, and it is plain that the method can be extended to the case of any number of masses situated on the axis of revolution.

This mode of obtaining a graphic representation of the lines of force is given by Clerk Maxwell, "Electricity and Magnetism," Part I., Chapter vii.

35. **Force acting on Element of Surface of charged Conductor.**—When a conductor is charged with electricity in equilibrium, the electricity in each element of its surface is, in general, acted on by a force normal to the

element; and as the electricity cannot leave the element, the same force tends to move the element itself relatively to the rest of the surface, and produces stresses if this motion be prevented by the cohesion of the material of the conductor. The electric force just outside the element dS is $4\pi\sigma$, and inside the surface, in the substance of the conductor, is zero.

To determine the force J which would act on an element of surface dS if it contained a unit of electric mass, imagine a surface S_1 described in the substance of the conductor, inside and close to S. Since, by Article 30, the resultant force at each point of this surface is zero, $\int NdS_1 = 0$; and therefore, by Art. 26, since there is no mass on S_1, the total mass inside it must be zero. Now imagine another surface S_2 coinciding with S_1 except at the element dS where it coincides with S; then $\int NdS_2 = 4\pi M + 2\pi M'$; but by what precedes $M = 0$, and $M' = \sigma dS$; also, N is zero everywhere on S_2 except in the element dS, where it is the same as J; we have therefore $JdS = 2\pi\sigma\, dS$, that is, $J = 2\pi\sigma$. The force J is the force per unit of electric mass acting in the element. To get the mechanical force FdS tending to move the element, or producing stress in the material of the conductor, we must multiply J by the electric mass in the element, that is, by σdS. Hence we obtain

$$FdS = 2\pi\sigma^2 dS = \frac{R^2}{8\pi}\, dS,$$

where R is the resultant force just outside the element; whence F, the mechanical force acting on the unit of area of the surface of the conductor, is given by the equation

$$F = \frac{R^2}{8\pi}. \qquad (8)$$

The value of J can be obtained in another way by considering the normal force immediately inside and immediately outside dS. The total normal force immediately outside dS is made up of two parts, one due to the distribution on dS itself, which may be denoted by N_1, and one due to the electricity on the rest of the conductor, which may be denoted by N_2. Inside dS the corresponding normal forces may be

denoted by N'_1 and N'_2; then $N'_1 + N'_2 = 0$, and by (5), Art. 29, $N_1 + N_2 = 4\pi\sigma$; but N_2 and N'_2 can differ only by an infinitely small quantity; also $N'_1 = - N_1$; hence $N_2 = N_1 = 2\pi\sigma$. Again, the total resultant force which the electricity on the element dS exerts on itself must be zero, and therefore $J = N_2 = 2\pi\sigma$.

36. **Uniplanar Distribution.** — In the case of a uniplanar distribution, the force varying inversely as the distance, the results of the preceding Articles hold good with some slight modifications.

In this case curves take the place of surfaces, and solid angles are replaced by plane angles, so that 2π expresses the total angle round a point.

Thus Gauss' Theorem, Art. 26, becomes

$$\int N ds = 2\pi M + \pi M'. \tag{9}$$

Instead of equation (4) we have

$$R_2 d_2 = R_1 d_1 + 2\pi M, \tag{10}$$

where d_1 and d_2 are orthogonal diameters of an infinitely small uniplanar tube of force.

For (5) we must substitute

$$R_2 \cos\psi_2 = R_1 \cos\psi_1 + 2\pi v, \tag{11}$$

where v is the uniplanar line density on the curve s. Hence, when P passes perpendicularly through the curve s, the normal component of the resultant force at P in the direction in which P moves is increased by $2\pi v$. Equations (6) and (7) become

$$l(X_2 - X_1) + m(Y_2 - Y_1) = 2\pi v, \quad \lambda(X_2 - X_1) + \mu(Y_2 - Y_1) = 0. \tag{12}$$

To see that these equations are consistent with those for cylindrical distributions in three dimensions, the force varying inversely as the square of the distance, we must remember that, in a uniplanar, distribution, the mass contained in an element of area is double the mass in the cylinder of unit height standing on this area in the corresponding cylindrical distribution, but that the resultant force R acting on the unit of mass is the same for the two distributions.

We see now that, in a uniplanar distribution, the force acting on the mass contained in an element of area is double the force acting on the mass in the corresponding element of volume in the cylindrical distribution. A similar result holds good for an element of a curve and the corresponding element of the surface of a cylinder. Thus instead of equation (8) we have in the case of a uniplanar distribution

$$\mathsf{F} = \frac{R^2}{4\pi}, \tag{13}$$

where F is the force per unit of length acting on a charged curve.

As a uniplanar distribution is merely a mathematical artifice, (13) must be transformed into (8) in order to have a physical meaning.

For a uniplanar distribution, tubes of equal induction due to a single mass situated at a point A are sectors of a circle having equal angles with A as centre. Hence the method given in Art. 34 for drawing lines of force due to the joint action of any number of isolated masses, is readily applicable to a uniplanar distribution, the equations given in that Article for determining the tubes of equal induction being replaced by

$$m'\theta'_1 = i, \quad \theta'_2 = 2\theta'_1, \quad \theta'_3 = 3\theta'_1, \text{ \&c.}$$

37. **Equality of Total Mass in Equivalent Distributions.**—If two different distributions of mass produce the same normal force at every point of a closed surface S surrounding both, the integral $\int NdS$ is the same for one distribution as for the other, and therefore, Art. 26, $M_1 = M_2$, where M_1 is the total mass in the first distribution, and M_2 the total mass in the second.

Again, if the resultant force at each point of S be the same for the two distributions, N must be the same, and therefore as before $M_1 = M_2$.

If the resultant force for the two distributions be the same at all points outside S, it must be the same at each point of S, and therefore we conclude that—

If two distributions produce the same resultant force at

every point of space external to a closed surface S surrounding both, the total mass in the one distribution must be equal to the total mass in the other.

If a curve be substituted for a surface, it is obvious that the results of this Article hold good for a uniplanar distribution in which the force varies inversely as the distance.

38. **Centrobaric Distribution.**—If a distribution of mass M be such that its resultant force in external space always passes through the same point O, the distribution is said to be *centrobaric*, and the point O is called the *baric centre*.

When a distribution is centrobaric, the resultant force at any point P in external space is the same as if the entire mass M were concentrated at the baric centre O.

To prove this, since the lines of force are straight lines passing through O, a tube of force must be a cone having its vertex at O; then if $d\omega$ denote the solid angle of this cone, Σ its section, and R_1 the resultant force at a point P whose distance from O is r, we have, by Art. 28, along this tube

$$R_1 r^2 d\omega = R_1\Sigma = \text{constant} = m_1 d\omega, \text{ whence } R_1 = \frac{m_1}{r^2};$$

and in like manner the resultant force R_n at any point on a line L_n of force is given by the equation

$$R_n = \frac{m_n}{r^2}.$$

Now suppose two points P and Q at the same distance r from O, at which the forces are $\frac{m_1}{r^2}$ and $\frac{m_2}{r^2}$; take on OP and OQ points P' and Q' such that $PP' = QQ'$, and connect PQ and $P'Q'$ by arcs of circles having O as centre; then the work done by the forces of the field on an element of mass in moving round the closed circuit $PQQ'P'P$ is zero; but along the circular arcs there is no work done, as they are perpendicular to the lines of force. Hence, in going from r to r', the work done by the force $\frac{m_1}{r^2}$ is equal to the work

done by the force $\frac{m_2}{r^2}$, and therefore $m_1 = m_2$: in general, therefore, $R = \frac{m}{r^2}$; but this is the expression for the force produced by a mass m placed at O; hence, by Art. 37, we have $m = M$, and the theorem is proved.

If there be an unoccupied region $\mathfrak{S}$ of space in the interior of a distribution of mass M which is centrobaric for this region, in which the baric centre O is situated, so that a closed surface S can be drawn round O having no mass in its interior; then at every point of $\mathfrak{S}$ the resultant force is zero.

For by a mode of procedure similar to that employed above, it can be shown that, at any point of the unoccupied region whose distance from O is r, the resultant force is $\frac{m}{r^2}$. Hence for the surface S, by Art. 26, we have

$$\int N dS = 4\pi m,$$

but again, since there is no mass inside S, we must have

$$\int N dS = 0; \text{ whence } m = 0;$$

and therefore at every point of the unoccupied region the resultant force is zero.

In a centrobaric distribution due to mass within a finite distance from the origin, the baric centre must coincide with the centre of mass. This follows from the consideration that, at a point at infinity, the forces due to the various mass elements act in parallel lines, and are proportional to the corresponding masses. Hence their resultant passes through the centre of mass; and therefore this point must be the intersection of two such resultants, and consequently must coincide with the baric centre.

The theorems relating to the baric centre which have been established above may be proved in a similar manner for a centrobaric distribution of uniplanar mass for which each element of force varies inversely as the distance.

Examples.

1. Determine the lines of force and the equipotential surfaces in the interior of a homogeneous ellipsoid.

The differential equations of a line of force are

$$\frac{dx}{Ax} = \frac{dy}{By} = \frac{dz}{Cz},$$

where A, B, C are the same as in (15), Art. 21.

Integrating these equations we have

$$z^{\frac{1}{C}} = K_1 x^{\frac{1}{A}} = K_2 y^{\frac{1}{B}},$$

where K_1 and K_2 are arbitrary constants.

The differential equation of an equipotential surface is

$$Ax\,dx + By\,dy + Cz\,dz = 0,$$

which integrated becomes

$$Ax^2 + By^2 + Cz^2 = K,$$

where K is an arbitrary constant.

2. The attraction of a homogeneous spherical shell at an internal point is zero, and at an external point is the same as if its entire mass were concentrated at its centre.

It is obvious from symmetry that the shell is centrobaric both for an internal and an external point, the baric centre being the centre of the sphere. Hence by Art. 38 we obtain the above results.

3. If the resultant force in unoccupied space be uniform in direction it must be uniform in magnitude.

In this case the lines of force are parallel straight lines; a tube of force is therefore a cylinder whose section is constant.

Hence by Art. 28 the force is constant along any one line of force; and by a method similar to that employed in Art. 38, we can show that the force does not vary in going from one line of force to another.

In this case the equipotential surfaces are planes.

4. A hollow closed conductor is charged with electricity in equilibrium. If there be no mass in the interior hollow, show that at every point in it the resultant force is zero, and that there is no charge at any point on the inner surface of the conductor.

Since the interior hollow $\mathfrak{S}$ is unoccupied, no tube of force can begin or end there; and therefore, since a tube of force cannot be a closed curve, if there be a tnbe of force in $\mathfrak{S}$, it must begin and end on the interior surface of the conductor, which by Art. 31 is impossible.

Again, since there is no force on either side of the interior surface of the conductor, by Art. 29 there can be no charge.

5. Show that the electricity on a charged insulated ellipsoidal conductor is in equilibrium when it is so distributed that its density at each point of the external surface of the ellipsoid is proportional to the normal thickness of the coincident homœoid.

If the electricity be distributed in this manner, by Art. 18 the force at a point in the substance of the conductor is zero; and therefore, by Art. 29, at the external surface, the resultant force is in the direction of the normal. The conditions for electric equilibrium are therefore, Art. 30, fulfilled.

6. In Ex. 5 prove that σ the electric density at any point Q of the surface of the conductor, R the resultant force at a point outside the surface and infinitely near Q, and F the force per unit of area acting in the surface of the conductor at Q, are given by the equations

$$\sigma = \frac{Ep}{4\pi abc}, \quad R = \frac{Ep}{abc}, \quad F = \frac{1}{8\pi}\,\frac{E^2 p^2}{a^2 b^2 c^2},$$

where E is the total charge on the conductor, a, b, c its semi-axes, and p the perpendicular from its centre on the tangent plane at Q.

By Ex. 4, Art. 24, we have $\sigma = \kappa p$, where κ is constant, but $\int \sigma dS = E$, and $\int p dS = 4\pi abc$; hence we obtain σ, and the expressions for R and F follow from Arts. 29 and 35.

From the expressions for σ, R, and F, it follows that these quantities become very large at points near the extremities of an elongated ellipsoidal conductor for which b and c are very small compared with a.

7. Determine the distribution of electricity on an ellipsoidal conductor when one axis becomes evanescent.

If p be the central perpendicular on the tangent plane at the point x, y, z on the surface, we have, in general,

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4};$$

whence

$$\frac{c^2}{p^2} = c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) + \frac{z^2}{c^2};$$

when z and c are each zero, this becomes

$$\frac{c^2}{p^2} = \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Hence

$$\frac{p}{c} = \frac{ab}{\sqrt{(a^2 b^2 - b^2 x^2 - a^2 y^2)}},$$

and

$$\sigma = \frac{E}{4\pi\sqrt{(a^2 b^2 - b^2 x^2 - a^2 y^2)}}.$$

It is plain that the ellipsoid is transformed into an elliptic plate whose thickness at the edge is an infinitely small quantity of the second order. The density at the edge is infinite, but on that part of the surface of the plate where the density is infinite the total mass is infinitely small.

In fact, at a point on the ellipse whose semi-axes are νa and νb, we have

$$b^2 x^2 + a^2 y^2 = \nu^2 a^2 b^2,$$

and therefore

$$\sigma = \frac{E}{4\pi ab\sqrt{1-\nu^2}}.$$

It is now easy to show, by integration, that on one surface of the plate between this ellipse and the edge the total amount of mass is $\frac{1}{2}E\sqrt{1-\nu^2}$, which is zero when $\nu = 1$.

It is scarcely necessary to remark that a conductor such as that here supposed could not be realized in nature.

If the ellipsoid be of revolution it becomes a circular plate, and we have

$$\sigma = \frac{E}{4\pi a\sqrt{(a^2 - r^2)}},$$

where r is the distance from the centre of the point of the plate where the surface density is σ.

8. Find a distribution of electricity consistent with equilibrium on a charged insulated infinitely thin elliptic plate.

The distribution σ, which has been obtained in Ex. 7 for the elliptic plate A into which an ellipsoid with an evanescent axis is transformed, gives at each point of the plate not on its edge a force perpendicular to its plane. If we suppose an infinitely thin plate B, whose thickness follows a different law from that of A, but whose elliptic boundary is the same, to be substituted for A, the force at any point of B, not on its edge, due to the distribution σ on its surface can differ by only an infinitely small quantity from the force at the corresponding point of A, and is therefore normal to B. Hence electricity E distributed over an infinitely thin elliptic plate, whose thickness follows any law, is in equilibrium if its density σ on each surface of the plate at the point x, y be given by the equation

$$\sigma = \frac{E}{4\pi\sqrt{\{a^2 b^2 - b^2 x^2 - a^2 y^2\}}}.$$

The thickness of the plate B at its edge may be an infinitely small quantity of the *first* order, but the total mass distributed over the surface intercepted between the two faces of B at its edge remains the same as the corresponding mass for A, and is therefore infinitely small, though the mode of its distribution is undetermined. Accordingly the expression obtained above for σ remains valid.

In a subsequent chapter it will be proved that, under given conditions, there is only one possible distribution of electricity consistent with equilibrium.

9. A spherical soap-bubble is charged with a quantity E of electricity; if this charge be just sufficient to keep the bubble in equilibrium, the air inside and outside being at the same pressure, prove that

$$16\pi a T = \frac{E^2}{a^2},$$

where a is the radius of the soap-bubble, and T the tension along a great circle across the unit of length of the perpendicular arc.

It is here supposed that the material surface of the sphere is equally expanded in all directions so that the tension is the same along all great circles. The normal pressure at a point P on the sphere is equilibrated by the tensions along all the great circles passing through P. Consider now a small circle having its pole at P, the arc ds of a great circle from P to its circumference being infinitely small. If $d\tau$ be the angle between two tangents to a great circle passing through P at the points where it meets the small circle, we have $a d\tau = 2ds$. Again the resultant of the two tensions acting at P along the same great circle is $2T\,ds\,d\theta \sin \frac{1}{2}d\tau$. Hence the resultant of all the tensions passing through P is

$$\pi T d\tau\, ds \quad \text{that is} \quad 2\pi \frac{T}{a} ds^2,$$

and this must equilibrate the total force acting normally to the spherical area enclosed by the small circle having ds for radius. Hence if F be the magnitude of this force per unit of area,

$$\pi F ds^2 = 2\pi \frac{T}{a} ds^2.$$

Substituting for F from Art. 35, and remembering that $\sigma = \dfrac{E}{4\pi a^2}$, we obtain the equation required.

10. Prove that at any point of the surface of an ellipsoidal mass of homogeneous liquid, rotating in relative equilibrium round a permanent axis, the force acting on a fluid particle is proportional and parallel to the corresponding semi-diameter of the reciprocal ellipsoid.

The components of the force are

$$-(A-\omega^2)x, \quad -(B-\omega^2)y, \quad \text{and} \quad -Cz,$$

whence, by Ex. 6, Art. 24, the above result is obvious.

11. Show that, for any distribution of mass, every line of force which does not encounter mass or pass through a point of equilibrium has an asymptote passing through the centre of mass.

From Art. 31 it appears that the line of force has a point at infinity: and, as the resultant force at this point passes through the centre of mass, the truth of the theorem is obvious.

12. Find the equation of a line of force due to masses m', m'', m''', &c., situated at points on the same straight line.

By Art 34 we have

$$I = 2\pi \{m'(1-\cos\theta') + m''(1-\cos\theta'') + \&c.\};$$

whence

$$m'\cos\theta' + m''\cos\theta'' + \&c. = M - \frac{I}{2\pi},$$

where M is the sum of the masses. If I be constant, this equation represents a line of force.

It can readily be obtained from the consideration that the component of the resultant force perpendicular to a line of force at any point on this line is zero; hence

$$\frac{m'}{r'^2}\frac{r'd\theta'}{ds} + \frac{m''}{r''^2}\frac{r''d\theta''}{ds} + \&c. = 0;$$

but $$r' \sin\theta' = r'' \sin\theta'' = r''' \sin\theta''' = \&c. = \&c.;$$

whence $$m' \sin\theta'\, d\theta' + m'' \sin\theta''\, d\theta'' + \&c. = 0,$$

which, by integration, gives the equation above.

13. In Ex. 12 show how to draw the asymptote to a line of force for which I is given.

The asymptote passes through the centre of mass, and makes with the axis an angle θ given by the equation

$$\cos\theta = 1 - \frac{I}{2\pi M}.$$

If $I > 4\pi M$, the value obtained for $\cos\theta$ is impossible. In this case, the corresponding line of force has no point at infinity, and must therefore start from one of the acting masses and end on another, or pass through a point of equilibrium.

If all the masses have the same algebraical sign, there is no line of force for which $I > 4\pi M$; for if there were, we should have for a point on this line

$$-2\pi(m' \cos\theta' + m'' \cos\theta'' + \&c.) > 2\pi(m' + m'' + \&c.),$$

which, on the hypothesis above, is impossible.

14. When the sum of the masses is zero, no line of force at a finite distance from the axis can extend to infinity.

15. For two masses whose magnitudes are equal, and whose algebraical signs are opposite, the lines of force are magnetic curves. See Ex. 17, Art. 17.

16. In the case of a uniplanar distribution, the lines of force for two equal masses, one attractive, the other repulsive, are circles.

17. For two equal repulsive masses the lines of force in a uniplanar distribution are hyperbolas.

18. If a field of force result from the action of masses whose sum is zero, and which are situated at a finite distance from the origin, no tube of force for which the corresponding induction is not infinitely small can extend to infinity.

If such a tube extended to infinity, the induction over the portion of the sphere at infinity intercepted by it would be a quantity not infinitely small; but when the sum of the acting masses is zero, so also is the induction over the whole sphere at infinity, and the induction over any portion of this sphere cannot differ from zero by more than an infinitely small quantity.

CHAPTER IV.

THE POTENTIAL.

SECTION I.—*Elementary Properties.*

39. **Historical.**—When a particle, whose mass is unity, is moving under the action of a force whose components X, Y, Z at any point are functions of its coordinates, the velocity v of the particle is given by the equation

$$v^2 = 2\int(Xdx + Ydy + Zdz) + C.$$

If the quantity under the integral sign is a perfect differential we obtain, by integration, a function U of the coordinates which is called the *force function.*

Laplace seems to have been the first to employ the force function in the solution of questions relating to attractions. The powerful analysis of Laplace led to many results of great value, and showed the importance of a study of the properties of this function. The research was taken up with splendid success by Green, whose great theorem may be said to dominate the whole field of the higher Mathematical Physics, and to whom the term Potential is due.

Lagrange, in the "Mécanique Analytique," made frequent use of the force function which corresponds to a material system and which is obtained by integration from that belonging to a particle; but in his Equations of Motion in generalized coordinates he substituted another function which is equal to the force function with its sign changed. This latter function has an important physical meaning, as it expresses the potential energy of the moving system. To it, therefore, the term potential can be applied more properly than to the force function.

40. **Definition of the Potential.**—*The potential of a mass system at any point* is the energy due to the mutual action of the unit of mass placed at the point, and the system, regarded as invariably connected, placed in its actual position.

The potential of a system of repelling or attracting masses at any point may, perhaps, be more simply defined as *the work done against the forces of the field, due to the system, in bringing the unit of mass to the point from an infinite distance, the positions of the acting masses being supposed invariable.*

Since no energy is expended in bringing an invariable system *unacted on by external force* into any assigned position, it is plain that one of these definitions is equivalent to the other.

When we have to do with attractive masses the potential, as defined above, is negative. Hence, in the case of a gravitation potential, if the acting masses be regarded as positive, it is simpler to define the potential, at any point, as the work done by the forces of the system in bringing the unit of mass from an infinite distance to the point. The potential is then the same as the force function, and is the function which was employed by Laplace. In questions relating to gravitation the potential is still commonly so regarded.

It would seem to be the simplest method in all cases to adopt as the definition of the potential the second of those given above; and if the acting mass be attractive to regard it as negative, the unit mass acted on being always positive. In this way the algebraical expression for the potential will be the same as that for the function used by Laplace; and when we have to consider only the potential and the resultant force, the same algebraical formulæ will be equally valid for electric and gravitational masses. It must, however, be remembered that there is no mathematical artifice by which one algebraical formula, interpreted in the same manner, can be made to express the two physical facts, that masses of like kind repel one another in the case of electrical action, and that they attract one another in the case of gravitation.

When the distribution of mass is *cylindrical*, the potential, as defined above, is infinite, and this is true also for the corresponding uniplanar distribution.

For such a distribution, if X and Y be the components of the resultant force at any point of the uniplanar field, the potential V may be defined by the equation $V = -\int (X\,dx + Y\,dy)$, where no constant is to be added. This definition is provisional only, and is to be regarded as relative to the mode of arriving at the form of the potential given in the next Article.

41. **Mathematical Expression for the Potential.**—The force exerted at any point P, whose distance from the origin is r, by a mass m situated at the origin, is $\frac{m}{r^2}$; and the work done by this force in moving the unit mass from P to an infinite distance is

$$\int_r^\infty \frac{m}{r^2}\,dr, \text{ that is, } \frac{m}{r}.$$

Hence the potential at P of a mass m at any point Q is $\frac{m}{\mathrm{r}}$, where r is the distance QP.

Since the work done by any set of forces in a given displacement is the sum of the works done by the different forces taken separately, if V denote the potential at P due to any system of masses m_1, m_2, &c., whose distances from P are r_1, r_2, &c., we have

$$V = \Sigma\frac{m}{\mathrm{r}}. \tag{1}$$

When the acting mass is continuously distributed, this equation becomes

$$V = \int\frac{dm}{\mathrm{r}}. \tag{2}$$

In the case of a uniplanar distribution, the force due to a mass m at the origin is $\frac{m}{r}$, also $\int\frac{m\,dr}{r} = m\log r$; hence, according to the definition at the end of Article 40, the potential V for a uniplanar distribution of mass m is given by the equation

$$V = \Sigma m \log\frac{1}{\mathrm{r}}. \tag{3}$$

When the unit of length is assigned, this equation gives a definite value for the potential at every point.

42. **Spherical Shell and Sphere.**—To calculate the potential of a homogeneous thin spherical shell at a point P whose distance from the centre C of the shell is c. In this case the element of mass is $2\pi\sigma\, a^2 \sin\phi\, d\phi$, where a denotes

the radius of the sphere, and ϕ the angle which the radius drawn to any point of its surface makes with CP; taking P for origin, we have

$$V = 2\pi\sigma \int \frac{a^2 \sin\phi \, d\phi}{r} = \frac{2\pi\sigma a}{c} \int dr,$$

since $$r^2 = a^2 + c^2 - 2ac \cos\phi.$$

If P be outside the sphere, the limits of the integral are $c + a$ and $c - a$; if P be inside, they are $a + c$ and $a - c$. Hence for an external point $V = \dfrac{4\pi\sigma a^2}{c}$, and for an internal $V = 4\pi\sigma a$. If now we suppose the point P variable, and take C for origin, we have

$$V = \frac{4\pi\sigma a^2}{r}, \quad \text{or} \quad V = 4\pi\sigma a,$$

according as P is external or internal.

If P be on the surface of the sphere, the value of V for one form of the function is the same as for the other.

From the equations for V given above, we learn that the potential of a homogeneous spherical shell has the same constant value for all points inside it; and for all external points is the same as if the entire mass of the shell were concentrated at its centre.

From this last result we may conclude that the potential of a homogeneous solid sphere at a point P outside it is given by the equation $V = \dfrac{M}{r}$, where M is the mass of the sphere, and r the distance of P from its centre. The same equation holds good if the sphere be composed of homogeneous layers, comprised between spheres concentric with the external surface.

The potential of a homogeneous solid sphere at an internal point P, whose distance from the centre C is r, is the sum of the potential of the concentric sphere passing through P and of the thick shell comprised between this sphere and the external boundary. If the radius of this latter be a,

and that of any of the spheres making up the shell be R, we have, therefore,

$$V=\frac{4}{3}\pi\rho r^2+4\pi\rho\int_r^a RdR=2\pi\rho a^2-\frac{2}{3}\pi\rho r^2.$$

Here again we see that, at the external surface of the solid sphere, the two forms of the function V have the same value, viz., $\frac{4}{3}\pi\rho a^2$.

Examples.

1. Find the potential of a uniform thin bar of density λ and length l at any point P.

Take the middle point of the bar for origin, let ξ and η be the coordinates of P, and x the coordinate of any point on the bar, its line of direction being taken as the axis of x; then, if V be the potential of the bar at P, we have

$$V=\lambda\int\frac{dx}{\sqrt{\{(x-\xi)^2+\eta^2\}}}=\lambda\log\{x-\xi+\sqrt{\{(x-\xi)^2+\eta^2\}}\}$$

between $x=-\frac{1}{2}l$ and $x=\frac{1}{2}l$. If r and r' be the distances of P from the extremities of the bar,

$$V=\lambda\log\frac{r'+\frac{1}{2}l-\xi}{r-\frac{1}{2}l-\xi}.$$

To express V in terms of r and r' we have

$$r^2-r'^2=(\xi+\tfrac{1}{2}l)^2-(\xi-\tfrac{1}{2}l)^2=2l\xi;$$

whence, substituting for ξ, we get

$$V=\lambda\log\frac{(r'+l)^2-r^2}{r'^2-(r-l)^2}=\lambda\log\frac{(r+r'+l)(r'+l-r)}{(r+r'-l)(r'+l-r)}=\lambda\log\frac{r+r'+l}{r+r'-l}.$$

2. Find the potential at a point P of the portion of a homogeneous spherical shell intercepted between two planes perpendicular to the line joining P to the centre of the sphere.

If V be the potential required, a the radius of the sphere, c the distance of its centre from P, and σ the density of the shell, then

$$V=\frac{2\pi\sigma a}{c}\int dr=\frac{2\pi\sigma a}{c}(r_2-r_1),$$

where r_2 and r_1 are the distances from P of points on the small circles which are the boundaries of the annulus.

If M be the mass of the annulus, since

$$M=\frac{\pi\sigma a}{c}(r_2^2-r_1^2),\quad\text{we have}\quad V=\frac{2M}{r_2+r_1}.$$

3. Mass acting inversely as the square of the distance is uniformly distributed on the circumference of a circle; prove that the chord of contact of tangents drawn from an external point P divides the mass into two parts having equal potentials at P.

The normals to the circle at the two points at which it is met by a line drawn through P make equal angles with this line. Hence, the potentials of the two portions of mass are composed of elements which are equal respectively.

4. Find the uniplanar potential of a homogeneous circle at any point P, the force varying inversely as the distance.

Let v denote the uniplanar density of the circle, a its radius, c the distance of P from its centre O; then if Q be any point on its circumference, and if r and ϕ denote the distance PQ and the angle POQ, the potential V is given by the equation

$$V = av\int_0^{2\pi} \log\left(\frac{1}{r}\right) d\phi.$$

Now $$r^2 = a^2 + c^2 - 2ac\cos\phi = c^2\left(1 - \frac{a}{c}e^{i\phi}\right)\left(1 - \frac{a}{c}e^{-i\phi}\right),$$

whence, if P be external to the circle,

$$\log r = \log c - \tfrac{1}{2}\left\{\frac{a}{c}\left(e^{i\phi} + e^{-i\phi}\right) + \tfrac{1}{2}\frac{a^2}{c^2}\left(e^{2i\phi} + e^{-2i\phi}\right) + \&c.\right\}$$

$$= \log c - \left\{\frac{a}{c}\cos\phi + \frac{a^2}{2c^2}\cos 2\phi + \&c.\right\}, \text{ and therefore } \int_0^{2\pi}\log r\, d\phi = 2\pi\log c.$$

Substituting in the expression for V, we get $V = 2\pi va\log\dfrac{1}{c}$. If P be internal, we obtain, in a similar manner,

$$\int_0^{2\pi}\log r\, d\phi = 2\pi\log a,$$

whence $$V = 2\pi va\log\frac{1}{a}.$$

5. Find an expression for the potential V of a plane lamina, of uniform density σ, at any point O, the force varying inversely as the square of the distance.

Take O for origin, let c be the perpendicular distance of O from the plane of the lamina, N the foot of this perpendicular, and ϖ the distance of any point in the lamina from N; then the element of mass is expressed by $\sigma\varpi\, d\varpi\, d\phi$, and, as $r^2 = \varpi^2 + c^2$, we have

$$V = \iint\frac{\sigma r\, dr\, d\phi}{r} = \sigma\int(r - c)\, d\phi,$$

where the integral is to be taken round the curve bounding the lamina. If N be inside this curve, $V = \sigma\left\{\int r\, d\phi - 2\pi c\right\}$, and if outside, $V = \sigma\int r\, d\phi$.

The original expression for V may be transformed in another manner, as follows:

$$\int(r - c)\, d\phi = \int\left(\frac{r^2}{r} - c\right)d\phi = \int\frac{\varpi^2 + c^2 - rc}{r}\, d\phi = \int\frac{\varpi^2\, d\phi}{r} - c\int\frac{r - c}{r}\, d\phi.$$

Now if p be the perpendicular from N on a tangent to the curve s bounding the lamina, $\varpi^2 d\phi = p\,ds$; again, if ω be the solid angle subtended at O by the lamina, and θ the angle which r makes with ON, we have

$$d\omega = \frac{\varpi\, d\varpi\, d\phi}{r^2}\cos\theta = \frac{r\,dr\,d\phi}{r^2}\,\frac{c}{r};$$

whence
$$\omega = c\int\left(\frac{1}{c}-\frac{1}{r}\right)d\phi = \int\frac{r-c}{r}\,d\phi,$$

and therefore, by substitution,

$$V = \sigma\left\{\int\frac{p\,ds}{r} - c\omega\right\}.$$

If the lamina be bounded by straight lines, $\int\frac{p\,ds}{r}$ for one side of the lamina becomes $p_1\int\frac{ds}{r}$, or $p_1 v_1$, if v_1 denote the potential at O of this side regarded as having a density equal to unity. Hence

$$V = \sigma\{p_1 v_1 + p_2 v_2 + \text{\&c.} - c\omega\},$$

where p_1, p_2, &c. are the perpendiculars from N on the sides of the lamina, and v_1, v_2, &c., their potentials at O.

6. Find the potential of a homogeneous solid figure, bounded by plane faces, at any point O.

If ρ denote the density, and V the potential at O, of the solid, we have

$$V = \iiint\frac{\rho\, r^2\,dr\,d\omega}{r} = \rho\iint\frac{r^2}{2}\,d\omega.$$

The volume of the elementary cone having O for vertex, and standing on the element dS which subtends the solid angle $d\omega$ at O is expressed by $\frac{r^3}{3}\,d\omega$, and also by $\frac{\varpi\,dS}{3}$, where ϖ is the perpendicular from O on dS. Hence, if ϖ_1, ϖ_2, &c. denote the perpendiculars from O on the faces of the polyhedron, and v_1, v_2, &c. the potentials at O of these faces regarded as of unit density, we have

$$V = \frac{\rho}{2}\left\{\varpi_1\int\frac{dS_1}{r} + \varpi_2\int\frac{dS_2}{r} + \text{\&c.}\right\} = \frac{\rho}{2}\{\varpi_1 v_1 + \varpi_2 v_2 + \text{\&c.}\}.$$

In this equation the perpendiculars ϖ_1, &c. are to be regarded as positive or negative according as the radii vectores corresponding to each pass out of or into the polyhedron after meeting the face to which they are drawn.

7. Find the component in any given direction of the attraction of a homogeneous polyhedron at the origin O.

If the polyhedron be on the positive side of the origin, and x, y, z denote the coordinates of any point in it, then X, the required component of attraction, is given by the equation

$$X = \iiint\frac{\rho\,dx\,dy\,dz}{r^2}\,\frac{dr}{dx} = -\rho\iint dy\,dz\left(\frac{1}{r_2}-\frac{1}{r_1}\right) = -\rho\,\Sigma v\cos\alpha,$$

where α is the angle which the normal, drawn outwards at any point of the

surface of the polyhedron, makes with the axis of x, and v is the potential at O of the corresponding face regarded as of unit density.

The same result can be obtained from the expression for V in Ex. 6.

In this case V is a force function, and if we change the origin and take ξ, η, ζ as the coordinates of O, we have

$$X = \frac{dV}{d\xi} = \frac{\rho}{2}\left\{\Sigma v \frac{d\varpi}{d\xi} + \Sigma\varpi \frac{dv}{d\xi}\right\};$$

also $$\frac{d\varpi}{d\xi} = -\cos\alpha, \quad \frac{dv}{d\xi} = -\iint \frac{dS}{r^2}\frac{dr}{d\xi} = \iint \frac{dS}{r^2}\frac{dr}{dx} = \iint \frac{dS}{r^2}\cos\theta,$$

and $$\frac{\varpi}{r} = \cos\psi,$$

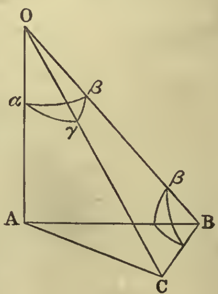
where ψ and θ are the angles which r makes with the normal and with the axis of x. Substituting, we have

$$X = -\frac{\rho}{2}\Sigma v \cos\alpha + \frac{\rho}{2}\iint \cos\theta \cos\psi \frac{dS}{r};$$

but $$\iint \cos\theta\cos\psi\frac{dS}{r} = \iint \frac{\cos\theta\, r^2\, d\omega}{r} = \iiint \frac{\cos\theta\, r^2\, dr\, d\omega}{r^2} = \iiint \frac{dx\,dy\,dz}{r^2}\frac{dr}{dx}$$

$$= -\iint \frac{dy\,dz}{r} = -\iint \frac{\cos\alpha\, dS}{r}.$$

8. Find the components of the attraction of a homogeneous right-angled triangle at a point O on the perpendicular to the plane of the triangle at one of its acute angles.



Let ABC be the triangle, C being the right angle, and let O be on the perpendicular to its plane through A; let $AB = c$, $BC = a$, $CA = b$, $OA = p$, $OB = r_2$, $OC = r_3$, and let α, β, γ denote the angles of the spherical triangle in which the planes meeting at O intersect a sphere of unit radius having O as centre; then, if X, Y, Z be the components of the attraction at O parallel to AC, CB, and OA, we have

$$X = \sigma\iint \frac{dx\,dy}{r^2}\frac{dr}{dx} = -\sigma\int\frac{dy}{r} = \sigma(v_c \sin A - v_a),$$

where v_c and v_a denote the potentials at O of the lines AB and BC, regarded as of unit density. In like manner

$$Y = \sigma(v_b - v_c \sin B),$$

and by Ex. 19, Art. 17, we have $Z = \sigma\omega$, where ω is the solid angle which the triangle ABC subtends at O. Since CB is perpendicular to the plane ACO, the angle γ is right, and as $\omega = \alpha + \beta + \gamma - \pi$, we have $\omega = \beta - \left(\frac{\pi}{2} - \alpha\right)$; but $\alpha = A$, and $\tan\alpha = \frac{a}{b}$.

Again, if we describe a sphere round B as centre, we get from the spherical triangle formed by the planes meeting at B,

$$\sin OBA \tan\beta = \tan B, \quad \text{that is,} \quad \frac{p}{r_2}\tan\beta = \frac{b}{a};$$

we have, therefore,

$$\omega = \tan^{-1}\frac{r_2}{p}\frac{b}{a} - \tan^{-1}\frac{b}{a}.$$

By Ex. 1, we have

$$v_c = \log\frac{p + r_2 + c}{p + r_2 - c}, \qquad v_a = \log\frac{r_2 + r_3 + a}{r_2 + r_3 - a}, \qquad v_b = \log\frac{p + r_3 + b}{p + r_3 - b},$$

also, from the geometry of the figure,

$$r_2^2 = p^2 + c^2, \quad r_3^2 = r_2^2 - a^2 = p^2 + b^2;$$

hence

$$v_c = \log\frac{\sqrt{r_2 + c}\left(\sqrt{r_2 + c} + \sqrt{r_2 - c}\right)}{\sqrt{r_2 - c}\left(\sqrt{r_2 + c} + \sqrt{r_2 - c}\right)} = \log\frac{r_2 + c}{\sqrt{r_2^2 - c^2}}.$$

A similar process being applied in the case of v_a and v_b, we obtain

$$v_c = \log\frac{r_2 + c}{p}, \quad v_a = \log\frac{r_2 + a}{r_3}, \quad v_b = \log\frac{r_3 + b}{p}.$$

Substituting for v_a, v_b, v_c, and ω, the values found above, we get finally,

$$X = \sigma\left\{\frac{a}{c}\log\frac{r_2 + c}{p} - \log\frac{r_2 + a}{r_3}\right\},$$

$$Y = \sigma\left\{\log\frac{r_3 + b}{p} - \frac{b}{c}\log\frac{r_2 + c}{p}\right\},$$

$$Z = \sigma\left\{\tan^{-1}\frac{r_2}{p}\frac{b}{a} - \tan^{-1}\frac{b}{a}\right\},$$

with the equations

$$c^2 = a^2 + b^2, \quad r_2^2 = p^2 + a^2 + b^2, \quad r_3^2 = p^2 + b^2.$$

9. Find the attraction of a homogeneous plane polygon at any point O.

From O let fall a perpendicular OA on the plane of the polygon; from A let fall perpendiculars on each of the sides, and join A to each of the vertices. We have then a number of right-angled triangles having a common vertex at A; find by Ex. 8 the components of the attraction of each of these at O. The resultant of all these forces is the attraction required.

Most of the Examples in this Article are borrowed from Routh's "Analytical Statics," vol. ii.

43. **Differential Coefficients of the Potential.**—If the point P receive a displacement ds, the potential V becomes V' at the new position of P, and by its definition $V - V'$ is the work done by the forces of the field on the unit of mass in the displacement ds; but if F be the component of the resultant force in this direction this work is Fds; hence $Fds = -(V' - V)$, and therefore

$$F = -\frac{dV}{ds}. \tag{4}$$

If dx, dy, dz be the increments of the coordinates of P due to the displacement ds, we have

$$V' - V = \frac{dV}{dx}\,dx + \frac{dV}{dy}\,dy + \frac{dV}{dz}\,dz,$$

also
$$\frac{dx}{ds} = l, \quad \frac{dy}{ds} = m, \quad \frac{dz}{ds} = n,$$

where l, m, n are the direction cosines of ds; hence

$$F = -\left(l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz}\right). \tag{5}$$

As a particular case of the above equation, if X, Y, Z be the components of the resultant force along the axes,

$$-\frac{dV}{dx} = X, \quad -\frac{dV}{dy} = Y, \quad -\frac{dV}{dz} = Z. \tag{6}$$

It has been shown in Art. 15 that, for all continuous distributions of mass, through a volume, X, Y, Z are finite at every point of space. Hence we conclude that so also are the differential coefficients of V.

In the case of a surface distribution of mass, it has been shown in Art. 16 that at a point O on the surface the normal component of force is finite. That the two other components of force are finite may be shown in the following manner:

A tangential component of the attraction at O is made

up of two parts, of which one is due to a circular plate, of infinitely small radius a, having O for centre, and the other to the mass whose distance from O exceeds a.

Taking O for origin, the normal at O for the axis of z, and putting $x^2 + y^2 = p^2$, we have

$$X = \int_0^a \int_0^{2\pi} \frac{\sigma p \cos\phi \, dp \, d\phi}{p^2} + \iint \frac{\sigma}{\cos\psi} \frac{x}{r^3} \, dx dy,$$

where ψ denotes the angle which the normal to the surface at any point makes with the axis of z.

In the first integral

$$\sigma = \sigma_0 + \left(\frac{d\sigma}{dx}\right)_0 p \cos\phi + \left(\frac{d\sigma}{dy}\right)_0 p \sin\phi.$$

Hence the first integral is of the order a. In the second integral, z is a function of x and y, given by the equation of the surface; whence

$$\frac{d}{dx}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\left(\frac{dr}{dx} + \frac{dr}{dz}\frac{dz}{dx}\right):$$

therefore, denoting the second integral by X_2, and integrating by parts, we have

$$X_2 = -\int \frac{\sigma \, dy}{r \cos\psi} + \iint \left\{\frac{d}{dx}\left(\frac{\sigma}{\cos\psi}\right) - \frac{\sigma}{\cos\psi}\frac{z}{r^2}\frac{dz}{dx}\right\} \frac{dx \, dy}{r}.$$

The limiting curve next the origin, round which the single integral is to be taken, is a circle of radius a, at whose circumference r and $\cos\psi$ differ from a and unity respectively by infinitely small quantities of the second order; also $dy = a \cos\phi \, d\phi$, and σ differs from a constant by a quantity of the order a. Hence for this curve $\int \frac{\sigma \, dy}{r \cos\psi} = 0$.

Inside the sign of double integration $dx \, dy = p \, dp \, d\phi$, and when r is small, z is of the order r^2, and therefore the coefficient of $dp \, d\phi$ is always finite. Hence we conclude that X is finite.

When we have to do with a uniplanar distribution, it is plain from the definition of V that

$$-\frac{dV}{dx} = X, \quad -\frac{dV}{dy} = Y, \tag{7}$$

and, as X and Y are everywhere finite for a continuous distribution, here also we may conclude that the differential coefficients of V are finite throughout the whole plane.

In the case of a volume distribution, X, Y, Z are, Art. 15, not only finite but also continuous, and therefore their differential coefficients must be finite; whence we conclude that the second differential coefficients of V are finite. For a surface distribution, the normal component of force is discontinuous at the surface, and in this case, therefore, at the surface the second differential coefficients of V are infinite.

Similar results hold good for a uniplanar distribution. If it be *areal*, the second differential coefficients of V are finite everywhere in the plane, but if it be *linear* they are infinite at the curve on which the mass is distributed.

44. **Mathematical Characteristics of the Potential.**—For a continuous volume or surface distribution of finite mass the potential is *finite*, *continuous*, and *single valued* throughout the whole of space, as may be shown in the following manner:

If $d\mathfrak{S}$ denote an element of volume where the density of the acting mass is ρ, and r the distance of this element from the point P, the potential V at this point is given by the equation $V = \int \frac{\rho d\mathfrak{S}}{\mathrm{r}}$; this becomes, if P be taken for origin, $V = \int \rho r\, dr\, d\omega$, in which the quantities under the integral sign are always finite. Hence at all points, whether inside or outside the acting mass, V is finite if the total mass be so.

Again, if over a surface S there be a distribution whose density at the element dS is σ, the resulting potential V at any point P is given by the equation $V = \int \frac{\sigma dS}{\mathrm{r}}$. The surface element dS expressed in polar coordinates is given by the equation

$$dS = \sqrt{\{(r^2 \sin\theta\, d\theta\, d\phi)^2 + (r \sin\theta\, dr\, d\phi)^2 + (r\, dr\, d\theta)^2\}}.$$

Hence the expression for dS contains r as a factor, and, if P be taken for origin, the quantity under the sign of integration in the expression for V remains finite when r is zero. Accordingly V is finite.

When the acting mass is finite, and at a finite distance from the origin, the potential at infinity is obviously infinitely small.

Since by the last Article the differential coefficients of V are everywhere finite, we conclude that V is continuous.

The continuity of V appears also from the consideration that the contribution of an element of mass to the value of V at the point P is infinitely small, even if P be inside the acting mass.

A function of the coordinates of a point is single valued when it has only one value for given values of the coordinates. Such are all algebraical functions consisting of integer powers or of fractional powers, expressing real quantities of given algebraical sign. On the other hand, an inverse trigonometrical function, such as $\tan^{-1}\frac{y}{x}$, is many valued, that is, for any given values of x and y the function admits of an infinite number of values.

The algebraical expression for the potential at a point P shows that it is a single valued function of the coordinates of P.

That V is single valued may be arrived at indirectly if we consider that, in the displacement of the unit of mass round a closed circuit, the work done by the forces of the field must be zero; because, if it were not, work could be obtained from permanent natural agents without loss of energy or consumption of material. Hence we conclude that for every closed circuit $\int\frac{dV}{ds}\,ds$ is zero, and therefore that the value of V at any point is independent of the path by which the point is reached.

It may happen that, at each point of a certain region $\mathfrak{S}$ of space, $V=\phi$, where ϕ is not a single valued function; but if this be so, there must be a region adjacent to $\mathfrak{S}$ in which V is not equal to ϕ, or else a surface S situated in $\mathfrak{S}$ in passing through which V changes discontinuously. This

latter alternative cannot, as we saw above, hold good when V is due to a continuous distribution of electric or gravitating mass, but may be fulfilled in the case of magnetic forces. Whichever alternative be the true one, $\int \frac{d\phi}{ds} ds = 0$ for every closed circuit which can be drawn without passing out of $\mathfrak{S}$ or cutting the surface S, and it is only for such a circuit that V is continuously equal to ϕ, and for which therefore the equation

$$\int \frac{dV}{ds} ds = \int \frac{d\phi}{ds} ds$$

is valid. The truth of this last statement is obvious when the circuit lies partly in a region for which V is not equal to ϕ: in the other case, that is, if there be a surface S in passing through which V changes discontinuously, if ϕ_1 and ϕ_2 be the values of ϕ which are equal to V at the points P_1 and P_2 at opposite sides of this surface, ϕ_2 is not the value of ϕ consecutive to ϕ_1, but is consecutive to ϕ'_1, a value of ϕ at P_1 differing from ϕ_1 by a finite amount; for example, if ϕ were an angle, we should have $\phi'_1 = \phi_1 \pm 2\pi$; thus, for a circuit cutting the surface S, the potential V is not continuously equal to ϕ.

By regarding the surface S as a boundary to $\mathfrak{S}$, even though $\mathfrak{S}$ is on both sides of it, we may consider that any circuit cutting S does not lie inside $\mathfrak{S}$, and say that, in any case, for every closed circuit or cycle inside $\mathfrak{S}$ we must have $\int \frac{d\phi}{ds} ds = 0$. Hence ϕ may be said to be *acyclic* for the region $\mathfrak{S}$.

We conclude that if for any region of space V is equal to a function ϕ which is not a single valued function, then ϕ must be *acyclic* throughout this region.

The expressions for the potential of a homogeneous spherical thin shell and of a homogenous sphere given in Art. 42 illustrate the statement that the potential V at a point P is finite, continuous, and single valued whatever be the position of P. From Art. 42 we learn also that, in the case of a sphere, as P passes from space

occupied by mass into space unoccupied, the form of V, regarded as a function of the coordinates of P, changes. That this is true in general will be shown in Art. 45.

In the case of a uniplanar areal distribution, the potential V at any point at a finite distance from the acting mass, if this point be taken for origin, may be expressed by the equation $V = -\int \tau r \log r \, dr \, d\theta$. Here the coefficient of $dr\, d\theta$ sign is zero when r is zero, and is always finite so long as r is finite.

When the uniplanar distribution is linear, since the element of the curve on which the uniplanar mass is distributed is expressed by

$$\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{\frac{1}{2}} dr,$$

the value of the potential V at the origin is given by the equation

$$V = -\int_0^{r_1} v \log r \, dr - \int_0^{r_2} v \log r \, dr + \&c.,$$

where the terms under the integral sign, not written down, have r^2 as a factor, and vanish therefore at the origin. Integrating the first two terms by parts, we get

$$V = -\Big|_0^{r_1} v\,(r \log r - r) - \Big|_0^{r_2} v\,(r \log r - r)$$

$$+ \int_0^{r_1} (r \log r - r)\, dv + \int_0^{r_2} (r \log r - r)\, dv + \&c.$$

Here, when $r = 0$, the corresponding terms outside the sign of integration vanish, and so also does the coefficient of dv under the sign of integration; whence it appears that, if the origin be on the curve where there is mass, V remains finite.

Again, as the contribution to the value of V afforded by the element of mass at the origin is infinitely small, V is continuous.

At a point P at an infinite distance, the potential of a uniplanar distribution of mass is $\Sigma m \log \frac{1}{r}$; if R be the

distance of P from the centre of mass, which, except $\Sigma m = 0$, is at a finite distance from all points of the acting mass, we have $\mathrm{r} = R + f$, where f is finite; then

$$\frac{1}{\mathrm{r}} = \frac{1}{R}\left(1 - \frac{f}{R} + \&\text{c.}\right);$$

whence $$V = \Sigma m \left\{\log \frac{1}{R} + \log\left(1 - \frac{f}{R} + \&\text{c.}\right)\right\},$$

which differs by only an infinitely small quantity from $(\Sigma m) \log \frac{1}{R}$. Unless Σm be zero, the value obtained for V is infinite and of a sign opposite to that of the total mass.

When the total uniplanar mass is zero, its potential at infinity is an infinitely small quantity of the order $\frac{a}{R}$, where a is finite, and R is the distance of a point at infinity from a point in the acting mass.

This appears from the consideration that, if the total mass be zero, for every positive element m of mass there must be an equal negative element; then if R be the distance of the former, and $R + a$ that of the latter, from a point P at infinity, the joint potential of the two elements of mass at P is v, where

$$v = m \log \frac{R + a}{R} = m \log\left(1 + \frac{a}{R}\right) = m\left(\frac{a}{R} + \&\text{c.}\right),$$

which is of the order $\frac{a}{R}$; hence the whole potential at P is of this order.

That V is single valued can be shown for a uniplanar in the same manner as for a three-dimensional distribution.

On the whole, therefore, we conclude that, for a continuous distribution of finite uniplanar mass comprised within a finite area, V is everywhere continuous and single valued; and is finite at all points within a finite distance of the acting mass, but infinite at an infinite distance, unless the total mass be zero, in which case V is zero.

45. **Equations of Laplace and Poisson.**—If we integrate the quantity

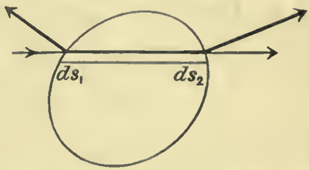
$$\left(\frac{d^2V}{dx^2} + \frac{d^2V}{dy_2} + \frac{d^2V}{dz^2}\right) d\mathfrak{S},$$

where $d\mathfrak{S}$ denotes an element of volume, throughout the region bounded by the closed surface S, we obtain

$$\iiint\left(\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}\right) dx\,dy\,dz$$

$$= \iint\frac{dV}{dx}\,dy\,dz + \iint\frac{dV}{dy}\,dz\,dx + \iint\frac{dV}{dz}\,dx\,dy.$$

The double integral



$$\iint\frac{dV}{dz}\,dx\,dy$$

denotes the sum, for the entire surface S, of the quantities

$$\left\{\left(\frac{dV}{dz}\right)_2 - \left(\frac{dV}{dz}\right)_1\right\} dx\,dy,$$

where $\left(\frac{dV}{dz}\right)_1$ and $\left(\frac{dV}{dz}\right)_2$ are the values of $\frac{dV}{dz}$ at the points where a line parallel to the axis of Z enters and leaves the space bounded by the surface S. If l, m, n be the direction cosines of the normal to the surface *drawn outwards*, and dS_1 and dS_2 the elements of surface intercepted by the prism whose section is $dx\,dy$, we have $-n_1\,dS_1 = dx\,dy = n_2\,dS_2$; whence

$$\left\{\left(\frac{dV}{dz}\right)_2 - \left(\frac{dV}{dz}\right)_1\right\} dx\,dy = n_1\left(\frac{dV}{dz}\right)_1 dS_1 + n_2\left(\frac{dV}{dz}\right)_2 dS_2,$$

and therefore

$$\iint\frac{dV}{dz}\,dx\,dy = \int n\,\frac{dV}{dz}\,dS.$$

If we proceed in a similar manner with the two other double integrals, we get

$$\iiint\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)dx\,dy\,dz$$

$$=\int\left(l\frac{dV}{dx}+m\,\frac{dV}{dy}+n\,\frac{dV}{dz}\right)dS=-\int NdS,$$

where N is the component normal to the surface of the resultant force. By Gauss' Theorem, Art. 26, we have

$$\int NdS=4\pi M;$$

and if M be due to a contiuous volume distribution of density ρ, we have $M=\int\rho d\mathfrak{S}$, whence

$$\int\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}+4\pi\rho\right)d\mathfrak{S}=0.$$

Since this equation is true for any volume, however small, it is true for the element $d\mathfrak{S}$; and therefore, in space occupied by mass, whose density at the point x, y, z is ρ, we have

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}+4\pi\rho=0. \qquad (8)$$

This is known as Poisson's equation.

In space unoccupied by mass, (8) becomes

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0, \qquad (9)$$

which is the equation of Laplace.

The operator

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$$

is of such frequent occurrence in Mathematical Physics that it is convenient to indicate it by a distinct symbol.

The notation which will be adopted in the present treatise is that of Thomson and Tait, *Natural Philosophy*, and of Williamson, *Integral Calculus*, in accordance with which

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$$

will be denoted by the symbol ∇^2.

Clerk Maxwell uses ∇ to express the quaternion operator

$$i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz},$$

so that, with him, ∇^2 denotes

$$-\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right).$$

Theoretically speaking, this notation is more perfect than the other; but when there is no reference to quaternions, the introduction of the negative sign is inconvenient.

We may now write the equations of Laplace and Poisson in the form

$$\nabla^2 V=0, \quad (10) \qquad\qquad \nabla^2 V+4\pi\rho=0. \quad (11)$$

As the point x, y, z passes from occupied into unoccupied space, V remains finite and continuous, but there is an abrupt change in the value of $\nabla^2 V$. From this we learn that the form of V, regarded as a function of x, y, z, cannot be the same in the two regions of space.

At a surface on which there is a distribution of finite mass, neither of the above equations holds good. In fact in this case $\nabla^2 V$ is infinite and indeterminate.

46. **Differential Equation for the Potential at a charged Surface.**—If P be a point on a surface at which the surface density is σ, and N and N' be the normal components at this point of the resultant force on the sides A and A' of the surface in the direction from A' to A, by Art. 29 we have $N=N'+4\pi\sigma$; now if ν and ν' be the normals drawn from P towards A and A', then

$$N=-\frac{dV}{d\nu},\ N'=\frac{dV}{d\nu'},$$

whence
$$\frac{dV}{d\nu} + \frac{dV}{d\nu'} + 4\pi\sigma = 0. \tag{12}$$

Equation (12) is called the characteristic equation at the charged surface. When the surface is closed, the form of V on one side is in general different from its form on the other.

Of this we have had an example in the case of a thin spherical shell, Art. 42.

47. **Differential Equations for Uniplanar Distribution.**—In the case of a uniplanar distribution, V is a function of x and y;

$$\text{then } \nabla^2 \text{ becomes } \frac{d^2}{dx^2} + \frac{d^2}{dy^2},$$

and $\iint \nabla^2 V\, dx\, dy$, taken through the space bounded by a closed curve s, is equal to $-\int N ds$ taken round the curve; this again, by Art. 36, equals $2\pi M$; whence, taking for s the boundary of an element where the uniplanar density is τ, we have

$$\nabla^2 V + 2\pi\tau = 0. \tag{13}$$

Again, by Art. 36, we see that at a curve on which there is a mass distribution of density υ, we have

$$\frac{dV}{d\nu} + \frac{dV}{d\nu'} + 2\pi\upsilon = 0. \tag{14}$$

48. **Transformation of Coordinates.**—It has been shown, in Art. 45, that

$$\int \nabla^2 V d\mathfrak{S} = \int \frac{dV}{d\mathrm{n}} dS, \tag{15}$$

where the first integral is taken through the volume bounded by the closed surface S, over which the second integral is taken, and n is the normal to the element dS drawn outward. From equation (15), we can readily deduce the form of $\nabla^2 V$ for any coordinates.

In the case of polar coordinates, we have

$$d\mathfrak{S} = r^2 \sin\theta\, dr\, d\theta\, d\phi;$$

and the surface elements of a sphere, of a right cone, and of a meridian plane, are, respectively, denoted by

$$r^2 \sin\theta \, d\theta \, d\phi, \qquad r \sin\theta \, dr \, d\phi, \qquad r \, dr \, d\theta.$$

Hence, if we integrate through the region enclosed by two spheres having the origin as centre, two coaxal cones having this point as vertex, and two meridian planes, by (15) we obtain

$$\iiint \nabla^2 \, V d\mathfrak{S} = \Big|_{r_1}^{r_2} \iint \frac{dV}{dr} r^2 \sin\theta \, d\theta \, d\phi$$

$$+ \Big|_{\theta_1}^{\theta_2} \iint \frac{dV}{rd\theta} r \sin\theta \, dr \, d\phi + \Big|_{\phi_1}^{\phi_2} \iint \frac{dV}{r \sin\theta d\phi} r \, dr \, d\theta$$

$$= \iiint \left\{ \sin\theta \frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) + \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) + \frac{1}{\sin\theta} \frac{d^2 V}{d\phi^2} \right\} dr \, d\theta \, d\phi.$$

If we now suppose the region through which the volume integral is taken to become infinitely small, we get

$$\nabla^2 \, V r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{1}{r^2} \left\{ \frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) + \frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) + \frac{1}{\sin^2\theta} \frac{d^2 V}{d\phi^2} \right\} r^2 \sin\theta \, dr \, d\theta \, d\phi \, ;$$

whence we obtain the operational equation

$$\nabla^2 = \frac{1}{r^2} \left\{ \frac{d}{dr} r^2 \frac{d}{dr} + \frac{1}{\sin\theta} \frac{d}{d\theta} \sin\theta \frac{d}{d\theta} + \frac{1}{\sin^2\theta} \frac{d^2}{d\phi^2} \right\}. \qquad (16)$$

If we put $\cos\theta = \mu$, we have

$$\frac{d}{d\theta} = - \sin\theta \frac{d}{d\mu}, \text{ and } \frac{1}{\sin\theta} \frac{d}{d\theta} = - \frac{d}{d\mu};$$

and, substituting from these equations, we obtain

$$\nabla^2 V = \frac{1}{r^2}\left\{\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) + \frac{d}{d\mu}(1-\mu^2)\frac{dV}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2V}{d\phi^2}\right\}. \quad (17)$$

On this mode of expressing $\nabla^2 V$ is based the spherical harmonic analysis of Laplace.

When the position of a point P is expressed by the perpendicular p let fall from it on a given line OZ, the distance z of the foot of this perpendicular from a given point O on this line, and the angle ϕ which a plane through OZP makes with a given plane through OZ, the quantities p, ϕ, z are called cylindrical coordinates.

In this case, $d\mathfrak{S} = p\,dp\,d\phi\,dz$, and the element of volume is bounded by the areas $p\,d\phi\,dz$, $dp\,dz$, $p\,dp\,d\phi$, and their opposites. By a process similar to that employed for polar coordinates we get then

$$\nabla^2 V = \frac{1}{p^2}\left\{p\frac{d}{dp}\left(p\frac{dV}{dp}\right) + \frac{d^2V}{d\phi^2}\right\} + \frac{d^2V}{dz^2}. \quad (18)$$

When V is a function of x and y only, we have

$$\nabla^2 V = \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2};$$

and, if we consider only the coordinate plane of x, y, the cylindrical coordinates p and ϕ become plane polar coordinates r and θ; whence, for the uniplanar potential, we obtain

$$\nabla^2 V = \frac{1}{r^2}\left\{\left(r\frac{d}{dr}\right)^2 + \frac{d^2}{d\theta^2}\right\}V. \quad (19)$$

Polar, or spherical, and cylindrical coordinates are particular cases of curvilinear coordinates.

For an account of the general method of curvilinear coordinates in which the position of a point is indicated by the intersection of three surfaces, and for the corresponding formula of transformation, see Williamson, *Trans.* R.I.A., vol. xxix. part xv.

49. **Law of Force in Electrical Action.**—It has been assumed in the preceding pages that the force between two elements of electric mass varies inversely as the square of the distance between them; but the most satisfactory method of proving this fact is somewhat indirect and depends on the determination of the form of the electric potential.

Whatever be the law of force for electric masses, the definition given in Art. 40 for the potential due to a mass system holds good, so that if $m\psi(r)$ denote the potential at a point P, whose distance from the origin is r, due to a mass m placed at the origin, the potential V at P of any mass system is given by the equation

$$V = \int \psi(\mathrm{r})\, dm, \tag{20}$$

where r is the distance of P from dm.

The force components at P are

$$-\frac{dV}{dx}, \quad -\frac{dV}{dy}, \quad \text{and} \quad -\frac{dV}{dz};$$

and if there be no force at any point of a given region, the potential throughout this region is constant.

It has been found experimentally, and the method is susceptible of great accuracy, that when an insulated conductor is charged with electricity in equilibrium there is no electric mass anywhere in its interior. The experiment can be performed by means of a conductor such that its outer layer of material can be separated into two portions and removed by means of insulating handles.

No matter what charge is originally imparted to the conductor, no electric mass can be detected in its interior portion after the outer layer has been removed. We conclude therefore that, in a charged insulated conductor, the whole of the electric mass is accumulated at the outside of the external surface.

Whatever be the distribution of electricity, or the law of force, there can be no force anywhere in *the substance* of a conductor in electric equilibrium, because if such a force existed it would produce a new distribution. Hence we conclude that the couche of electricity in equilibrium on the

external surface produces no force anywhere in the interior region, throughout which the potential, due to the surface distribution, is consequently constant.

If the charged conductor be a sphere, since this surface is perfectly symmetrical, the distribution of mass on it must be uniform, and therefore the law of force must be such that the potential of a homogeneous thin spherical shell is constant for all points in its interior.

Let a be the radius of the shell, σ its density, and r the distance of any point on its surface from a point P in its interior, whose distance from the centre is ξ, then the element of acting mass is found as in Art. 42; and from (20) we have for V, the potential at P, the equation

$$V = \frac{2\pi\sigma a}{\xi}\int_{a-\xi}^{a+\xi} r\psi(r)\,dr = 2\pi\sigma a\,\frac{f(a+\xi) - f(a-\xi)}{\xi}.$$

Hence $f(a+\xi) - f(a-\xi) = C\xi$, where C is constant. Differentiating twice with respect to ξ, we have

$$f''(a+\xi) - f''(a-\xi) = 0.$$

Hence $f''(r)$ must be independent of the value of r, in other words constant; whence $\dfrac{df}{dr} = C_1 r + C_2$, where C_1 and C_2 are constants; but $\dfrac{df}{dr} = r\psi(r) = rV$, and therefore $rV = C_1 r + C_2$, and $V = C_1 + \dfrac{C_2}{r}$, whence finally $-\dfrac{dV}{dr} = \dfrac{C_2}{r^2}$, that is, the law of force is that of the inverse square.

50. **Energy of a Mutually Repulsive System.**—When a system is composed of mutually repulsive mass, the internal repulsive forces tend to drive this mass asunder, and would in doing so, if not prevented by restraints or opposing forces, perform a certain amount of work which must be equal to the potential energy of the system in its actual state.

Let m denote the mass concentrated at any point Q of the system, and V the potential at that point. Imagine a system B geometrically identical with the given system A, but such that the mass at any point is equal to the mass at

the corresponding point of A multiplied by a quantity μ constant for the whole system. Then, since the distances are the same, and every mass in B is μ times the corresponding mass in A, the potential at any point in B is μ times the potential at the corresponding point of A.

Let us now suppose the element of mass $m_1 d\mu$ brought to the point Q_1 in B where the mass is μm_1, and the potential μV_1. The work required for this operation is $\mu V_1 m_1 d\mu$; and if a similar operation be performed for each point of the system B, the total work required is

$$(\mu V_1 m_1 + \mu V_2 m_2 + \&c.)\, d\mu.$$

Hence, if this work be denoted by dW, we have

$$dW = \mu d\mu\, \Sigma m V.$$

If μ be 1, the system B is identical with the given system A; and if we suppose μ to be increased continuously in the manner described above from 0 to 1, we obtain the total work required to bring together the system A. This work is equal to the potential energy W of the system; hence

$$W = \Sigma m V \int_0^1 \mu d\mu = \tfrac{1}{2} \Sigma m V. \qquad (21)$$

In the case of a mutually attractive system, it is plain that W expresses the work required to scatter the system to an infinite distance.

51. **Mutual Energy of two Invariable Systems.**—The energy due to the mutual action of two systems of mass when brought into any assigned relative position, each system being regarded as itself invariable, may be expressed by any one of three different forms.

If Q be any point of the first system, m the mass there concentrated, and V_P the potential of the whole system at any point P in space, and if Q', m', and V'_P have corresponding significations for the second system, the mutual energy W is given by the equations

$$W = \Sigma m' V_{Q'} = \Sigma m V'_Q = \Sigma \frac{mm'}{\mathrm{r}}, \qquad (22)$$

where r is the distance between Q and Q'.

The truth of the equations above is obvious from the definition of the potential.

52. **Change of Energy due to alteration of Mass.**—If the geometrical form of a system remain invariable, but the mass at each point be altered, the potential receives a corresponding change, and also the total potential energy due to the mutual action of the parts of the system on each other. If the mass at any point be changed from m to m', the potential at the same point from V to V', and the total energy of the system from W to W', we have

$$2(W' - W) = \Sigma m'V' - \Sigma mV.$$

If now we suppose the two systems in Art 51 to be geometrically coincident, by (22) we have

$$\Sigma mV' = \Sigma m'V;$$

whence

$$\Sigma m'V' - \Sigma mV = \Sigma(m' - m)(V' + V)$$

$$= \Sigma(m' + m)(V' - V);$$

and we get

$$W' - W = \tfrac{1}{2}\Sigma(m' - m)(V' + V) = \tfrac{1}{2}\Sigma(m' + m)(V' - V). \quad (23)$$

This equation can be established also in a manner similar to that employed in Article 50.

Examples.

1. Show that the component parallel to the axis of x of the repulsion of a homogeneous body, of density ρ, bounded by the surface S, at any external point O, is equal to the potential of a fictitious distribution on S whose density at any point of the surface is $l\rho$, where l is the cosine of the angle which the normal, drawn outwards at the point, makes with the axis of x.

Take O for origin, let X be the force component due to the repelling mass, and v the potential at O of the fictitious surface distribution; then

$$X = -\rho\iiint\frac{dx\,dy\,dz}{r^2}\frac{x}{r} = -\rho\iiint\frac{dx\,dy\,dz}{r^2}\frac{dr}{dx} = \rho\iint\frac{dy\,dz}{r} = \int\frac{\rho\,l\,dS}{r} = v.$$

A surface distribution of density $l\rho$ is merely a mathematical artifice, and physically impossible (see Art. 8).

2. Find the potential of a homogeneous thin circular plate at a point P on the perpendicular to its plane through its centre.

If ϖ be the distance of any point of the plate from its centre C, and c the distance PC, taking P for origin, we have $r\,dr = \varpi\,d\varpi$; whence

$$V = 2\pi\sigma \int_{r_0}^{r_1} dr = 2\pi\sigma\,(\sqrt{a^2+c^2} - c),$$

where a is the radius of the plate. If C be taken for origin, V assumes the form

$$2\pi\sigma\,(\sqrt{a^2+r^2} - r), \quad \text{or} \quad 2\pi\sigma\,(\sqrt{a^2+z^2} - z),$$

according as CP is regarded as a radius vector or a coordinate. In the latter case when z is negative V becomes $2\pi\sigma\,(\sqrt{a^2+z^2} + z)$.

3. Show that equation 22, Art. 24, follows immediately from Poisson's Equation.

4. Find the potential at any point of a field of force throughout which the resultant force is constant in direction.

Take a line parallel to this direction for axis of z. Since the lines of force are perpendicular to the equipotential surfaces, these latter are parallel planes. Hence when z is constant V is constant, that is, V is a function of z. Laplace's Equation therefore becomes

$$\frac{d^2V}{dz^2} = 0, \quad \text{whence} \quad V = C_1 z + C_2.$$

5. Find the potential at any point between two infinite parallel planes, each of which is at a constant potential.

Take as the plane of xy the plane whose potential is A, let b be the distance of the other, and B its potential; then V is plainly a function of z; and therefore, by Laplace's equation, we get

$$V = A - \frac{A-B}{b} z.$$

6. Two concentric spherical surfaces are each at a constant potential: find the potential at any point between them.

Let a denote the radius of the inner sphere, and b that of the outer, the potentials at their surfaces being A and B; then, if the centre of the spheres be taken for origin, it is plain that the potential at any point of the space between their surfaces is a function of r; and as this space is unoccupied, we have, by Laplace's equation,

$$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0;$$

whence
$$V = \frac{Bb - Aa}{b-a} + \frac{(A-B)\,ab}{b-a}\,\frac{1}{r}.$$

7. In the last example, if the spherical surfaces be the boundaries of two charged conductors in electric equilibrium, find the surface density and charge on each conductor.

If σ be the surface density at any point of the inner surface, and ν and ν' the normals to it drawn outwards and inwards,

$$\frac{dV}{d\nu} + \frac{dV}{d\nu'} + 4\pi\sigma = 0, \quad \text{but} \quad \frac{dV}{d\nu'} = 0,$$

since there is no force in the substance of the conductor; therefore

$$4\pi\sigma = -\frac{dV}{dr} = \frac{(A-B)\,ab}{b-a}\,\frac{1}{a^2},$$

which determines the surface density; also, if E be the total charge, we have

$$E = 4\pi\sigma a^2 = \frac{A-B}{b-a}\,ab.$$

If E' be the total charge on the outer spherical surface, by Art. 32

$$E' = -E.$$

This also appears directly from the value of $\frac{dV}{dr}$ at the outer surface.

A combination of two conductors, one of which entirely surrounds the other, constitutes what is termed a *condenser*. When the condenser is formed of two concentric spheres, it appears, from what is said above, that the charge on either surface is proportional to the difference of their potentials. That this is true, in general, will be proved subsequently. By means of an electric machine we can, in general, bring a conductor in communication with it to a given potential. The use of a condenser such as has been described is to enable us to increase the corresponding charge.

The charge on an insulated sphere at potential A is Aa; but when surrounded by another sphere, as described above, the charge on the inner sphere is $Aa\,\frac{b}{b-a}$ when the outer sphere is at potential zero, and the multiplier $\frac{b}{b-a}$ can be made very large.

8. In the case of a uniplanar distribution, find the potential of a homogeneous circular plate at any point in its plane.

Let a be the radius, and τ the uniplanar density of the plate; then, if the centre be taken for origin, V is plainly a function of r solely, and therefore

$$\nabla^2 V = \frac{1}{r}\,\frac{d}{dr}\left(r\frac{dV}{dr}\right).$$

At a point outside the plate, $\nabla^2 V = 0$; whence, by integration, $\frac{dV}{dr} = \frac{C_1}{r}$, and taken round the boundary of the plate

$$\int \frac{dV}{dr}\,ds = \int_0^{2\pi} C_1\,d\theta = 2\pi\,C_1;$$

but

$$-\int \frac{dV}{dr}\,ds = 2\pi M,$$

whence

$$C_1 = -M = -\pi\tau a^2, \quad \text{and} \quad V = \pi\tau a^2 \log\frac{1}{r},$$

no constant being added.

At a point inside the mass $\nabla^2 V + 2\pi\tau = 0$, from which, by integration,

$$\frac{dV}{dr} = -\pi\tau r + \frac{C}{r},$$

and, as there is no resultant force at the centre, where r is zero, C must be zero; and, therefore, integrating again, we have

$$V = -\frac{\pi\tau r^2}{2} + C_2.$$

At the boundary of the plate the external and internal values of V must be equal; whence

$$-\frac{\pi\tau a^2}{2} + C_2 = \pi\tau a^2 \log\frac{1}{a}.$$

Substituting for C_2 from this equation, we have, at an internal point,

$$V = \pi\tau\left\{\tfrac{1}{2}(a^2 - r^2) - a^2 \log a\right\}.$$

9. Two coaxal circular cylinders of infinite length are each at a constant potential; find the potential at any point between them.

Let a denote the radius of the inner cylinder, and b that of the outer, their potentials being A and B. Then, as the potential at any point between them is a function of p, its distance from the axis; and as this point is in unoccupied space, we have, by Laplace's equation,

$$\frac{1}{p}\frac{d}{dp}\left(p\frac{dV}{dp}\right) = 0,$$

integrating, we get $V = C_1 \log p + C_2$.

When $p = a$ the potential is A, and when $p = b$ the potential is B; whence we obtain

$$V = \frac{A - B}{\log b - \log a}\log\frac{1}{p} + \frac{A\log b - B\log a}{\log b - \log a}.$$

10. If the cylinders in the last example be the surfaces of conductors in equilibrium, find the density of the surface distribution on each.

If the surface densities be denoted by σ_1 and σ_2, we have

$$\sigma_1 = \frac{A - B}{4\pi a}\log\frac{b}{a}, \quad \sigma_2 = -\frac{A - B}{4\pi b}\log\frac{b}{a}.$$

From these expressions it appears that the charges on the two cylinders per unit of length are equal in magnitude, but opposite in algebraical sign.

11. Find the uniplanar potential of a circle on which there is a uniform linear distribution of mass.

If M denote the total uniplanar mass, a the radius of the circle, and r the distance of any point from its centre, the potential at an internal point is given by the equation

$$V = M\log\frac{1}{a},$$

and at an external by the equation

$$V = M\log\frac{1}{r}.$$

12. Show directly that if r be not zero,

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\frac{m}{\mathrm{r}} = 0,$$

where $$\mathrm{r}^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2.$$

13. How does an algebraical process, similar to the proof of the theorem above, fail when r is zero?

The algebraical expressions which were obtained for

$$\frac{d^2}{dx^2}\frac{m}{\mathrm{r}},\quad \frac{d^2}{dy^2}\frac{m}{\mathrm{r}},\quad \text{and}\quad \frac{d^2}{dz^2}\frac{m}{\mathrm{r}},$$

are then illusory, their real values being indeterminate as well as infinite.

14. If throughout any continuous region of unoccupied space the resultant force be constant in magnitude, show that it is constant in direction.

In this case $X^2 + Y^2 + Z^2 = $ constant, and therefore

$$\nabla^2 (X^2 + Y^2 + Z^2) = 0;$$

but $$\frac{d^2}{dx^2}X^2 = 2\left\{X\frac{d^2X}{dx^2} + \left(\frac{dX}{dx}\right)^2\right\}, \text{ \&c.},$$

whence, by addition,

$$X\nabla^2X + Y\nabla^2Y + Z\nabla^2Z + \left(\frac{dX}{dx}\right)^2 + \left(\frac{dX}{dy}\right)^2 + \left(\frac{dX}{dz}\right)^2$$

$$+ \left(\frac{dY}{dx}\right)^2 + \left(\frac{dY}{dy}\right)^2 + \left(\frac{dY}{dz}\right)^2 + \left(\frac{dZ}{dx}\right)^2 + \left(\frac{dZ}{dy}\right)^2 + \left(\frac{dZ}{dz}\right)^2 = 0;$$

now $$\nabla^2X = -\nabla^2\frac{dV}{dx} = -\frac{d}{dx}\nabla^2V = 0;$$

and in like manner, $\nabla^2Y = 0$, $\nabla^2Z = 0$; and therefore the sum of the nine squares $\left(\frac{dX}{dx}\right)^2$, &c. is zero: whence each of these squares is zero; and therefore $X = $ constant, $Y = $ constant, $Z = $ constant.

15. If the force due to an element of mass vary inversely as the n^{th} power of the distance, prove that the potential of a system of mass which is all of the same sign cannot be constant throughout a finite unoccupied portion of space except $n = 2$.

Throughout any portion of space where V is constant, ∇^2V must be zero; but if the force due to an element of mass m be $\frac{m}{\mathrm{r}^n}$, we have

$$\nabla^2V = (n-2)\,\Sigma\frac{m}{\mathrm{r}^{n+1}};$$

and when all the mass is of the same sign, this cannot be zero except $n = 2$.

53. **Potentials and Lines of Force for Uniplanar Distribution.**—In the case of an uniplanar distribution of mass acting inversely as the distance, the potential V in the unoccupied part of the plane satisfies the equation

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} = 0.$$

The general solution of the differential equation

$$\frac{d^2\phi}{dt^2} - a^2 \frac{d^2\phi}{dz^2} = 0$$

is readily obtained by assuming two new independent variables ξ and η connected with z and t by the equations

$$\xi = z + at, \qquad \eta = z - at,$$

from which we have

$$\frac{d}{dt} = a\frac{d}{d\xi} - a\frac{d}{d\eta}, \qquad \frac{d}{dz} = \frac{d}{d\xi} + \frac{d}{d\eta},$$

and therefore

$$4a^2 \frac{d^2\phi}{d\xi d\eta} = a^2 \frac{d^2\phi}{dz^2} - \frac{d^2\phi}{dt^2} = 0.$$

Hence

$$\phi = f_1(\xi) + f_2(\eta) = f_1(z + at) + f_2(z - at),$$

where f_1 and f_2 denote arbitrary functions.

If we now suppose $a^2 = -1$ the equation whose solution has been obtained becomes that satisfied by V; and putting $i = \sqrt{-1}$, we get

$$V = f_1(x + iy) + f_2(x - iy). \qquad (24)$$

Again, since $$\frac{d}{dx}\frac{dV}{dx} = -\frac{d}{dy}\frac{dV}{dy},$$

we may assume

$$\frac{dV}{dx} = \frac{d\psi}{dy}, \qquad \frac{dV}{dy} = -\frac{d\psi}{dx},$$

where ψ is a function of x and y. Hence, if x and y be the coordinates of a point on the curve ψ = constant, we get

$$\frac{dy}{dx} = \frac{\frac{dV}{dy}}{\frac{dV}{dx}},$$

and therefore the tangent to the curve at any point is in the direction of the resultant force at that point; accordingly ψ = constant is the equation of a line of force.

Again, $$d\psi = \frac{d\psi}{dx}\,dx + \frac{d\psi}{dy}\,dy$$

$$= -i\{f'_1(x+iy) - f'_2(x-iy)\}\,dx + \{f'_1(x+iy) + f'_2(x-iy)\}\,dy;$$

whence multiplying by i, and integrating, we get

$$i\psi = f_1(x+iy) - f_2(x-iy). \tag{25}$$

Putting $2f_1 = F$ we have from (24) and (25), by addition,

$$V + i\psi = F(x+iy). \tag{26}$$

Functions ϕ and ψ satisfying the equation

$$\phi + i\psi = F(x+iy) \tag{27}$$

are called conjugate functions of x and y. It is plain that if ξ and η be any two conjugate functions of x and y, the functions V and ψ are conjugate functions of ξ and η, and conversely that, if two functions ϕ and ψ are conjugate functions of ξ and η, they are conjugate functions of x and y.

Again, if ϕ and ψ be any two conjugate functions of x and y, we have, from (27), by differentiation,

$$\frac{d\phi}{dy} + i\,\frac{d\psi}{dy} = i\left(\frac{d\phi}{dx} + i\,\frac{d\psi}{dx}\right);$$

whence

$$\frac{d\phi}{dx} = \frac{d\psi}{dy}, \quad \text{and} \quad \frac{d\phi}{dy} = -\frac{d\psi}{dx},$$

and therefore

$$\frac{d^2\phi}{dx^2} = \frac{d^2\psi}{dxdy} = -\frac{d^2\phi}{dy^2},$$

that is, $\nabla^2\phi = 0$. Also, in like manner, $\nabla^2\psi = 0$.

A more complete account of the theory of conjugate functions, and of its application to the investigation of the potential and lines of force of a uniplanar distribution of mass, is reserved for a future chapter.

Examples.

1. Find the potential of a uniplanar distribution of mass whose lines of force are straight lines passing through a point O.

If r and θ be the polar coordinates of any point referred to O as origin, we have $e^{\log r + i\theta} = x + iy$. Hence, V and ψ are conjugate functions of log r and θ; but the equation of a straight line through O is $\theta =$ constant, and therefore

$$\psi = C\theta, \qquad V = C \log r,$$

where C is an undetermined constant.

2. If the lines of force of an uniplanar distribution of mass be confocal hyperbolas, find the potential.

If we assume

$$x = c \cosh \eta \cos \xi, \qquad y = c \sinh \eta \sin \xi,$$

we have

$$\frac{x^2}{c^2 \cos^2\xi} - \frac{y^2}{c^2 \sin^2\xi} = 1.$$

This equation represents a hyperbola whose primary semi-axis is $c \cos \xi$, and if ξ vary we obtain a system of confocal hyperbolas. Hence in this case the equation, $\xi =$ constant, represents a line of force.

Again, as

$$\cosh \eta = \cos i\eta, \quad i \sinh \eta = \sin i\eta,$$

we have

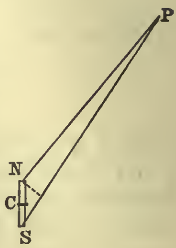
$$x + iy = c \cos (\xi - i\eta) = c \cos \left(\frac{\eta + i\xi}{i}\right).$$

Hence, η and ξ are conjugate functions of x and y, and we have

$$\psi = C\xi + C', \quad V = C\eta + C'',$$

where C, C', and C'' are undetermined constants.

54. **Potential of Magnetic Particle.**—Let S and N be the south and north poles, and C the centre of a magnetic particle, or small linear magnet, then the potential energy due to the presence of a unit north pole at a point P is the potential V of the particle at that point. Hence, if r_1 and r_2 denote the distances of P from S and N, and m the strength of the north pole of the particle, we have



$$V = \frac{m}{r_2} - \frac{m}{r_1} = \frac{m(r_1 - r_2)}{r_1 r_2}\ ;\ \text{but } r_1 - r_2 = ds \cos \varepsilon,$$

where ds is the length of the magnetic axis SN of the particle, and ε the angle it makes with CP, also the *magnetic moment* μ of the particle is defined by the equation $\mu = mds$, and the product $r_1 r_2$ differs by an infinitely small quantity, or, in the case of a small magnet by a negligible quantity, from r^2, where r is the distance of P from C; hence

$$V = \frac{\mu \cos \varepsilon}{r^2}. \tag{28}$$

If distances measured on lines parallel to the magnetic axis SN drawn through C and P be denoted by h' and h, and the distance CP be now denoted by r, the potential V may be expressed by either of the equations

$$V = \frac{d}{dh'} \frac{\mu}{\mathrm{r}} = -\frac{d}{dh} \frac{\mu}{\mathrm{r}}. \tag{29}$$

For any system of magnetic forces, the equation of Laplace holds good in unoccupied space; but in the case of a finite body continuously magnetized, as volume density has here no definite meaning, it is not obvious by what equation that of Poisson should be replaced.

As the potential due to magnetized bodies having finite dimensions requires special treatment, this subject is reserved for a future chapter.

55. **Magnetic Shell.**—A collection of magnetic particles so disposed that their centres form a continuous surface S to which their axes are normal is called a *magnetic shell*. The

sum of the magnetic moments of the particles whose centres lie on the surface element dS divided by that element is called the *strength of the shell*, and is supposed to be finite and continuous. If J denote the strength of the shell at any point, JdS may be regarded as the moment of a magnetic particle whose axis is the normal to dS.

If r denote the distance of dS from an external point O, at which V is the potential of the shell, ϵ the angle which r makes with the normal to dS, and $d\omega$ the element of solid angle which dS subtends at O, we have

$$V = \int J \frac{dS \cos \epsilon}{r^2} = \int J \frac{r^2 d\omega}{r^2} = \int J d\omega.$$

If the shell be uniform, J is constant, and

$$V = J\Omega, \tag{30}$$

where Ω is the solid angle which the shell, or its bounding curve, subtends at O.

In a magnetic shell whose surface is S, that side of S at which north poles of the magnetic particles are situated is regarded as the positive side.

Examples.

1. Prove that the potential at a point P of a uniform magnetic shell, whose strength is J, is increased by $4\pi J$ as P passes from the negative to the positive side of the shell.

The potential at P due to an element of the shell is $\frac{JdS \cos \epsilon}{r^2}$, but this is the expression for the normal force at P due to an element of surface dS whose density is J. As P passes through the element in the direction of the force, this force increases by $4\pi J$, Arts. 16, 29; therefore so also does the magnetic potential due to the same surface element of the shell; the rest of the magnetic potential varies continuously; hence, on the whole, the magnetic potential of the shell is increased by $4\pi J$ as P passes from the negative to the positive side of the shell.

2. Find the potential energy due to the mutual action of two small linear magnets. (See Art. 17.)

Take the centre of the first magnet for origin, and let h_1 and h_2 be lines parallel to the two magnetic axes, drawn through any point P, dh_1 and dh_2 being displacements in these directions; then, if ξ be a coordinate of P measured in any direction, $\frac{d\xi}{dh_1}$ is the cosine of the angle between the directions of ξ and h_1; also $\frac{dr}{dh_1}$ is the cosine of the angle between r and h_1. Similar results hold good for h_2.

Let V be the potential of the first magnet at the centre of the second, μ_1 the magnetic moment of the first, μ_2 that of the second, l_2 the length of its axis, and m_2 the strength of its north pole; then, if W denote the energy required, we have

$$W = -m_2\left(V - \tfrac{1}{2}\frac{dV}{dh_2}l_2\right) + m_2\left(V + \tfrac{1}{2}\frac{dV}{dh_2}l_2\right)$$

$$= m_2 l_2 \frac{dV}{dh_2} = \mu_2 \frac{dV}{dh_2}.$$

This may be put into two other forms. By Art. 54 we have

$$V = -\frac{d}{dh_1}\left(\frac{\mu_1}{r}\right);$$

whence

$$W = -\mu_1\mu_2 \frac{d^2}{dh_1 dh_2}\left(\frac{1}{r}\right).$$

Again, if ϵ_1 and ϵ_2 be the angles which r makes with h_1 and h_2, and θ_{12}, the angle between the two latter,

$$V = \frac{\mu_1 \cos\epsilon_1}{r^2} = \frac{\mu_1 r \cos\epsilon_1}{r^3};$$

and therefore

$$W = \mu_1\mu_2\left\{\frac{1}{r^3}\ \frac{d}{dh_2}(r\cos\epsilon_1) - \frac{3r\cos\epsilon_1}{r^4}\ \frac{dr}{dh_2}\right\}$$

$$= \frac{\mu_1\mu_2}{r^3}\{\cos\theta_{12} - 3\cos\epsilon_1\cos\epsilon_2\}.$$

In each of the following examples, two small linear magnets are supposed to act on each other.

3. Find the moment round the centre of the second magnet of the forces due to the action on it of the first.

Give the second magnet an angular displacement $d\phi$ round a perpendicular axis through its centre; if M be the component tending to increase ϕ of the required moment, the work done by the forces producing M in the displacement $d\phi$ is $Md\phi$, and this must be equal to the loss of potential energy, that is, to

$$-\frac{dW}{d\phi};$$

hence,

$$M = -\frac{dW}{d\phi} = -\frac{\mu_1\mu_2}{r^3}\left\{-\sin\theta_{12}\frac{d\theta_{12}}{d\phi} + 3\cos\epsilon_1\sin\epsilon_2\frac{d\epsilon_2}{d\phi}\right\},$$

since the displacement $d\phi$ does not alter r or ϵ_1.

If α and β be the angles which the plane of ϕ makes with the plane of h_1 and h_2, and the plane of r and h_2, respectively, from considering the arcs on a unit sphere described round the centre of the second magnet as centre, we have

$$d\theta_{12} = \cos\alpha\, d\phi, \quad d\epsilon_2 = \cos\beta\, d\phi,$$

whence

$$M = \frac{\mu_1\mu_2}{r^3}\sin\theta_{12}\cos\alpha - \frac{3\mu_1\mu_2}{r^3}\cos\epsilon_1\sin\epsilon_2\cos\beta.$$

This expression for M shows that it is the sum of the components in the plane of ϕ of two couples, one L_h in the plane parallel to the axes of the magnets, and one L_r in the plane of r and the axis of the second magnet, the magnitudes of these couples being given by the equations

$$L_h = \frac{\mu_1\mu_2}{r^3}\sin\theta_{12}, \quad L_r = -\frac{3\mu_1\mu_2}{r^3}\cos\epsilon_1\sin\epsilon_2.$$

The first couple tends to increase θ_{12}, and the second to diminish ϵ_2. The couple G whose moment is required is the resultant of these two.

4. Find the resultant of the forces exerted by the first magnet on the second.

Give the second magnet a displacement of translation parallel to a direction h_3; then if H_3 be the component of the required force tending to increase h_3, the work done by the forces exerted by the first magnet on the second in the displacement dh_3 is $H_3\,dh_3$, and this must be equal to the loss of potential energy, that is to $-\dfrac{dW}{dh_3}dh_3$;

hence, $$H_3 = -\frac{dW}{dh_3} = -\mu_1\mu_2\frac{d}{dh_3}\left\{\frac{\cos\theta_{12}}{r^3} - 3\frac{r\cos\epsilon_1\, r\cos\epsilon_2}{r^5}\right\}.$$

The angle θ_{12} is unaltered by a translation of the second magnet, and

$$\frac{dr}{dh_3} = \cos\epsilon_3, \quad \frac{d(r\cos\epsilon_1)}{dh_3} = \cos\theta_{13}, \quad \frac{d(r\cos\epsilon_2)}{dh_3} = \cos\theta_{23},$$

where ϵ_3, θ_{13}, θ_{23} are the angles which h_3 makes with r, h_1, and h_2, respectively. Hence, by substitution, we have

$$H_3 = \frac{3\mu_1\mu_2}{r^4}\{\cos\theta_{12}\cos\epsilon_3 + \cos\theta_{23}\cos\epsilon_1 + \cos\theta_{31}\cos\epsilon_2 - 5\cos\epsilon_1\cos\epsilon_2\cos\epsilon_3\}.$$

This expression shows that H_3 is the sum of the components along h_3 of three forces R, H_1, and H_2, in the directions of r, h_1 and h_2, respectively, the magnitudes of these forces being given by the equations

$$R = \frac{3\mu_1\mu_2}{r^4}(\cos\theta_{12} - 5\cos\epsilon_1\cos\epsilon_2),$$

$$H_1 = \frac{3\mu_1\mu_2}{r^4}\cos\epsilon_2, \quad H_2 = \frac{3\mu_1\mu_2}{r^4}\cos\epsilon_1.$$

The required force F is the resultant of these three.

If we suppose the couple G produced by two equal and opposite forces applied at the poles of the second magnet, one of these forces is of the order $F\dfrac{r}{l_2}$, and is therefore very great compared with F.

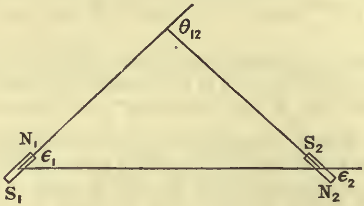
5. Find the couple and the force acting on the second magnet in each of the following cases:—

1°. When the axes of the magnets are in the line of centres.

2°. When the axes of the magnets are parallel to each other, and perpendicular to the line of centres.

3°. When the axis of the first magnet is in the line of centres and that of the second perpendicular to it.

4°. When the axis of the first magnet is perpendicular to the line of centres and the axis of the second in that line.



In estimating the angles h_1 is supposed fixed, and ϵ_1, ϵ_2, and θ_{12} are counted from it in the same direction; so that, if the three angles be in the same plane, $\theta_{12} = \epsilon_1 + \epsilon_2$.

1°. Here $\epsilon_1 = \epsilon_2 = \theta_{12} = 0$; whence $L_h = 0$, $L_r = 0$; also $\epsilon_3 = \theta_{13} = \theta_{23}$, and therefore

$$H_3 = -\frac{6\mu_1\mu_2}{r^4}\cos\epsilon_3;$$

whence F is an attraction along the line of centres whose magnitude is $\frac{6\mu_1\mu_2}{r^4}$.

2°. Here $$\epsilon_1 = \frac{\pi}{2}, \quad \epsilon_2 = -\frac{\pi}{2}, \quad \theta_{12} = 0;$$

whence $$L_h = 0, \; L_r = 0; \text{ also } H_3 = \frac{3\mu_1\mu_2}{r^4}\cos\epsilon_3;$$

and therefore F is a repulsion along the line of centres whose magnitude is

$$\frac{3\mu_1\mu_2}{r^4}.$$

3°. Here $$\epsilon_1 = 0, \quad \epsilon_2 = \theta_{12} = \frac{\pi}{2},$$

then, as L_h and L_r are in the same plane,

$$G = L_h + L_r = -\frac{2\mu_1\mu_2}{r^3}.$$

Again, $$R = 0, \quad H_1 = 0, \quad H_2 = \frac{3\mu_1\mu_2}{r^4}.$$

Hence the resultant couple tends to turn the second magnet into the line of centres, and the resultant force tends to move it in the direction of its own axis.

4°. Here $$\epsilon_1 = \frac{\pi}{2}, \quad \epsilon_2 = 0, \quad \theta_{12} = \frac{\pi}{2};$$

whence $$L_h = \frac{\mu_1\mu_2}{r_3}, \quad L_r = 0; \text{ also } R = 0, \quad H_1 = \frac{3\mu_1\mu_2}{r^4}, \quad H_2 = 0.$$

Hence the resultant couple tends to turn the axis of the second magnet into a direction opposite to that of the first, and the resultant force tends to move the second magnet in a direction parallel to the axis of the first.

It is to be observed that the couple in case 3° is double of the couple in case 4°. In case 3° the deflecting magnet is said to be *end on*, and the deflected magnet *broadside on*, whereas in case 4° the deflecting magnet is *broadside on*, and the deflected *end on*.

6. Find an expression for the energy due to the mutual action of two small magnets when the force emanating from a magnetic pole varies inversely as the n^{th} power of the distance.

In this case the potential of a magnet, whose moment is μ, at a point at a distance r from its centre is $\frac{\mu \cos \epsilon}{r^n}$, where ϵ is the angle which r makes with the magnetic axis. Then the mutual energy W of two small linear magnets whose magnetic moments are μ_1 and μ_2 is given by the equation

$$W = \frac{\mu_1 \mu_2}{r^{n+1}} \left\{ \cos \theta_{12} - (n+1) \cos \epsilon_1 \cos \epsilon_2 \right\}.$$

7. If the force due to a pole vary inversely as the n^{th} power of the distance, prove that when one small magnet is acted on by another, the deflecting couple when the deflector is end on and the deflected broadside on, is n times the deflecting couple when the deflector is broadside on and the deflected end on.

By a process similar to that employed in Ex. 5 the deflecting couple in the first case is found to be $-\frac{n\mu_1 \mu_2}{r^{n+1}}$, and in the second case $\frac{\mu_1 \mu_2}{r^{n+1}}$.

8. Prove that, in the two cases considered in Ex. 7, the resultant force on the deflected magnet has the same magnitude and direction, the latter being perpendicular to the line of centres.

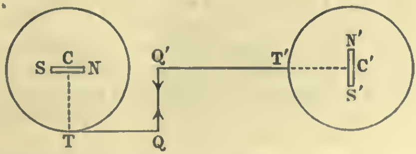
When the deflected magnet is broadside on,

$$R = 0, \quad H_1 = 0, \quad H_2 = \frac{(n+1)\mu_1 \mu_2}{r^{n+2}}.$$

When the deflected magnet is end on,

$$R = 0, \quad H_1 = \frac{(n+1)\mu_1 \mu_2}{r^{n+2}}, \quad H_2 = 0.$$

9. Show how to arrange an experiment to test the truth of the results obtained in cases 3° and 4° in Ex. 5.



Place two small magnets SN and $S'N'$, whose centres are C and C', in water on floats with rigid arms TQ and $T'Q'$ projecting from the floats, TQ being parallel to SN and perpendicular to CT, and $C'T'Q'$ perpendicular to

$S'N'$. If $C'Q'$ be made double of TQ, C placed on the production of $C'Q'$, TQ made parallel to $C'Q'$, and Q and Q' fastened together by a string equal in length to CT, the whole system is found to be in equilibrium.

This shows that the mutual action between the magnets when placed in the manner described is reducible to a single repulsive force perpendicular to the line of centres at the point of trisection of this line, the shorter segment being next the magnet which is end on.

This agrees with the results obtained in cases 3° and 4°, Ex. 5.

We see also from the reduction of the mutual action to a single force that the result in case 4°, Ex. 5, follows from that in case 3°, and that the couple acting on the magnet which is broadside on is double of the couple acting on the magnet which is end on.

The arrows in the figure indicate the directions of the tension of the string resisting at Q and Q' the forces due to the mutual action of the magnets.

10. If the law of the force due to a magnet pole be unknown, show how it may be determined.

The result of the experiment described in the last example combined with the theorem proved in Ex. 7 shows that the law of force must be that of the inverse square. For, by Ex. 7, if the force vary inversely as the n^{th} power of the distance in the experiment of Ex. 9, in order to have equilibrium, $C'Q'$ should be made n times TQ.

11. Prove that, when a small magnet M_2, whose centre is fixed, is in stable equilibrium under the action of another fixed small magnet M_1, the force produced by M_2 on a magnet pole at the centre of M_1 in the direction of its axis is the greatest possible.

The mutual energy W of the magnets is given by the equation

$$W = -\frac{d^2}{dh_1\,dh_2}\left(\frac{\mu_1\,\mu_2}{r}\right).$$

The centres of the magnets and the direction of h_1 being fixed when M_2 is in stable equilibrium, h_2 takes the direction which makes W a minimum. If we now suppose h_1 and h_2 to be drawn through C_1, the centre of M_1, instead of through C_2, the centre of M_2, the operators $\frac{d}{dh_1}$ and $\frac{d}{dh_2}$ both change sign, and the expression for W remains unaltered; therefore the direction of h_2 is such as to make $\frac{d}{dh_1}\,\frac{d}{dh_2}\left(\frac{1}{r}\right)$ a maximum; but if V_2 be the potential of M_2 at C_1, we have

$$\mu_2\frac{d}{dh_1}\,\frac{d}{dh_2}\left(\frac{1}{r}\right) = -\frac{d}{dh_1}\left(-\frac{d}{dh_2}\left(\frac{\mu_2}{r}\right)\right) = -\frac{dV_2}{dh_1}.$$

Hence the axis of M_2 is in such a direction as to make the force in the direction of h_1 exerted by M_2 on a magnet pole at C_1 the greatest possible.

The theorem of this example enables us to place a magnet with its centre at a given point so as to produce the greatest possible force in a given direction on a magnet pole placed at another given point.

12. When a small linear magnet M_2, whose centre C_2 is fixed, is in stable equilibrium under the action of another fixed magnet M_1, show that the axis of M_2 is in the direction of the force exerted by M_1 on a magnet pole at C_2, and that the axes of the two magnets lie in the same plane.

Let N_1 and S_1 be the poles of M_1, and N_2 and S_2 those of M_2, the length of its axis being l_2. The forces acting on M_2 at N_2 and S_2 are equal and opposite, neglecting quantities of the order $\frac{l_2}{r}$ as compared with these forces. Hence, in order that the moment round C_2 should vanish, S_2N_2 must be in the direction of the force on a north pole at C_2. The force exerted by the magnet M_1 on a pole at C_2 is the resultant of forces whose lines of direction are N_1C_2 and S_1C_2, and must therefore lie in the plane $C_2N_1S_1$.

13. The small linear magnet M_2, whose centre is fixed, is in equilibrium under the action of the fixed magnet M_1; find the resultant force on M_2.

By the last example the axes of M_2 and M_1 are in the same plane, and therefore $\theta_{12} = \epsilon_1 + \epsilon_2$; also in the equations of Ex. 3, $G = 0$; whence $\sin(\epsilon_1 + \epsilon_2) - 3\cos\epsilon_1 \sin\epsilon_2 = 0$; and therefore

$$\tan\epsilon_1 = 2\tan\epsilon_2, \quad \sin\epsilon_2 = \frac{\sin\epsilon_1}{\sqrt{(1 + 3\cos^2\epsilon_1)}}, \quad \cos\epsilon_2 = \frac{2\cos\epsilon_1}{\sqrt{(1 + 3\cos^2\epsilon_1)}};$$

then, from Ex. 4, we have

$$R = -\frac{3\mu_1\mu_2}{r^4}\,\frac{1 + 7\cos^2\epsilon_1}{\sqrt{(1 + 3\cos^2\epsilon_1)}},$$

$$H_1 = \frac{6\mu_1\mu_2}{r^4}\,\frac{\cos\epsilon_1}{\sqrt{(1 + 3\cos^2\epsilon_1)}}, \quad H_2 = \frac{3\mu_1\mu_2\cos\epsilon_1}{r^4}.$$

In this case, as the forces are all in one plane, we may resolve H_2 in the directions of R and H_1, and if R' and H'_1 denote the total forces acting on M_2 in the directions of r and h_1, we have

$$R' = -\frac{3\mu_1\mu_2}{r^4}\,\frac{1 + 4\cos^2\epsilon_1}{\sqrt{(1 + 3\cos^2\epsilon_1)}}, \quad H'_1 = \frac{3\mu_1\mu_2}{r^4}\,\frac{\cos\epsilon_1}{\sqrt{(1 + 3\cos^2\epsilon_1)}}.$$

14. In Ex. 13 show that the resultant force on M_2 tends to move it in the direction in which the force exerted by M_1 on a magnet pole at C_2, the centre of M_2, increases most rapidly.

Let V_1 be the potential of M_1 at C_2; then by Ex. 2,

$$W = \mu_2\frac{dV_1}{dh_2} = -\mu_2\sqrt{\left\{\left(\frac{dV_1}{dx}\right)^2 + \left(\frac{dV_1}{dy}\right)^2 + \left(\frac{dV_1}{dz}\right)^2\right\}},$$

since h_2 is in the direction of the resultant force on a magnet pole at C_2 by Ex. 12. The force on M_2 tends to move it in the direction in which W diminishes most rapidly; and from the expression for W given above, this must be the direction in which the force due to M_1 on a magnet pole at C_2 increases most rapidly in absolute magnitude.

SECTION II.—*General Theorems.*

56. **Gauss' Theorem.**—If V denote the potential at any point due to masses acting inversely as the square of the distance, and S the surface of a sphere whose radius is R, then

$$\int V dS = 4\pi R^2 U_0 + 4\pi RM, \tag{1}$$

where the integral is taken over the entire surface of the sphere, and where U_0 denotes the value at its centre of the potential of those masses which are outside it, and M the sum of those masses which are internal.

To prove this, let U denote the potential at any point due to the masses outside the sphere, and v that due to the masses which are internal, then $V = U + v$.

Now taking as origin the centre O of the sphere, by (2), Art. 26, we have

$$0 = \int \frac{dU}{dr} dS = R^2 \int \frac{dU}{dr} d\omega = R^2 \frac{d}{dr} \int U d\omega. \tag{2}$$

Hence $\int U d\omega$ taken over the surface of any sphere having O for centre is independent of the radius, provided the sphere does not enclose any of the mass producing U: therefore, by supposing the radius infinitely small, we obtain

$$\int U d\omega = 4\pi U_0.$$

Again, $$-4\pi M = \int \frac{dv}{dr} dS = R^2 \frac{d}{dr} \int v\, d\omega;$$

whence $$\frac{d}{dr} \int v\, d\omega = -\frac{4\pi M}{R^2};$$

and integrating on the hypothesis that the sphere is of variable radius, but continues to include the whole of the mass M, we have

$$\int v\, d\omega = 4\pi \frac{M}{R} + C.$$

If we suppose the radius R to become infinite, v is zero at each point of the surface of the sphere. Hence $C = 0$, and we get

$$\int v\,d\omega = 4\pi\frac{M}{R}.$$

Adding this equation to that previously obtained for U, we have

$$\int V\,d\omega = 4\pi\left\{U_0 + \frac{M}{R}\right\}, \tag{3}$$

from which (1) follows at once.

57. **Uniplanar Distribution.**—In the case of a uniplanar distribution of mass acting with a force varying inversely as the distance, Gauss' Theorem becomes

$$\int Vds = 2\pi R\;\left\{U_0 + M\log\frac{1}{R}\right\}, \tag{4}$$

where s is the arc, and R the radius of a circle, M the uniplanar mass inside it, and U_0 the potential at its centre of the external mass.

For in this case, at a point on the circle whose radius R is infinite, the value of v is $M\log\dfrac{1}{R}$ Hence the value of $\int v\,d\theta$ taken round this circle is $2\pi M\log\dfrac{1}{R}$, and the theorem can be proved in a manner similar to that employed above for a three-dimensional distribution of mass.

58. **Green's Theorem.**—If U and V be two functions of the coordinates which, as well as their differential coefficients, are acyclic, finite, and continuous throughout a region $\mathfrak{S}$ of space bounded by a surface S, then

$$\iint U\frac{dV}{d\nu}\,dS + \iiint U\nabla^2 V d\mathfrak{S} = \iint V\frac{dU}{d\nu}\,dS + \iiint V\nabla^2 U d\mathfrak{S}$$

$$= -\iiint\left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\frac{dV}{dz}\right)d\mathfrak{S}, \tag{5}$$

where the volume integrals are taken through the whole of

the region $\mathfrak{S}$, and the surface integrals over the whole of the boundary S, and ν denotes a normal to S drawn *into* the field of the triple integration.

To prove this, we integrate by parts the expression

$$\iiint \frac{dU}{dx}\,\frac{dV}{dx}\,dx\,dy\,dz,$$

and we get

$$\iiint \frac{dU}{dx}\frac{dV}{dx}\,dx\,dy\,dz = \iint U\frac{dV}{dx}\,dy\,dz - \iiint U\frac{d^2V}{dx^2}\,dx\,dy\,dz\,;$$

whence, by the addition of two similar equations, if Q denote

$$\iiint \left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\,\frac{dV}{dz}\right)d\mathfrak{S},$$

we have

$$Q = \iint U\left(\frac{dV}{dx}\,dy\,dz + \frac{dV}{dy}\,dz\,dx + \frac{dV}{dz}\,dx\,dy\right) - \iiint U\nabla^2 V d\mathfrak{S}.$$

By a process similar to that employed in Art. 45, we find that

$$\iint U\left(\frac{dV}{dx}\,dy\,dz + \frac{dV}{dy}\,dz\,dx + \frac{dV}{dz}\,dx\,dy\right) = -\iint U\,\frac{dV}{d\nu}\,dS\,;$$

whence, by substitution, we obtain

$$\iint U\frac{dV}{d\nu}\,dS + \iiint U\nabla^2 V d\mathfrak{S} = -\,Q. \qquad (6)$$

By an interchange of U and V we have, also,

$$\iint V\frac{dU}{d\nu}\,dS + \iiint V\nabla^2 U d\mathfrak{S} = -\,Q, \qquad (7)$$

and thus we get (5).

If the field $\mathfrak{S}$ consist of the space comprised between two closed surfaces of which one is inside the other, equation (5)

can be proved in the same manner as above, and this mode of proof is still valid when $\mathfrak{S}$ is the space outside any number of separate closed surfaces, and inside another enclosing them all, the surface integral in each of these cases being the sum of the surface integrals taken over each boundary. If a sphere whose radius is infinite be the outside surface, $\mathfrak{S}$ will consist of the whole of space outside a system of closed surfaces.

Again, if $\mathfrak{S}$ consist of a number of separate detached regions, Green's equation is obtained by the addition of the equations holding good for each of these regions respectively.

Lastly, if $\mathfrak{S}$ include the regions on both sides of a surface S, whether closed or open, Green's equation can still be proved in the same manner as before, the surface integral in this case being taken over both sides of the surface S.

When the region $\mathfrak{S}$ is bounded by a number of surfaces in any of the ways described above, Green's equation may be written in the form

$$\Sigma \iint U \frac{dV}{d\nu}\, dS + \iiint U\nabla^2 V d\mathfrak{S} = \Sigma \iint V \frac{dU}{d\nu}\, dS + \iiint V\nabla^2 U d\mathfrak{S}$$
$$= - \iiint \left(\frac{dU}{dx}\,\frac{dV}{dx} + \frac{dU}{dy}\,\frac{dV}{dy} + \frac{dU}{dz}\,\frac{dV}{dz} \right) d\mathfrak{S}; \qquad (8)$$

but the simpler form (5) is equally valid, provided it be understood that the surface integral is to be taken over the whole of all the boundaries.

When the functions U and V are identical, Green's equation becomes

$$\iint V \frac{dV}{d\nu} dS + \iiint V\nabla^2 V d\mathfrak{S} = - \iiint \left\{ \left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right\} d\mathfrak{S}. \qquad (9)$$

59. **Special Case of Green's Theorem.**—If the function U, or some of its differential coefficients, be infinite at a point P inside the field, the integrations implied by Green's equation may become illusory, and the equation itself may require modification.

If $U = U' + \dfrac{e}{r}$, where e is constant, r the distance of any point from the point P, and U' a function which is finite throughout the field, an important theorem can be obtained which is expressed by the equation

$$\iint U\frac{dV}{d\nu}dS + \iiint U\nabla^2 V d\mathfrak{S} = \iint V\frac{dU}{d\nu}dS - 4\pi e V_P + \iiint V\nabla^2 U' d\mathfrak{S}$$

$$= -\iiint\left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\frac{dV}{dz}\right)d\mathfrak{S}, \qquad (10)$$

where V_P is the value of V at the point P.

To prove this, describe, round the point P as centre, a sphere S' of radius a so small that S' does not meet any of the boundaries, and apply equation (5) to the field $\mathfrak{S}'$ consisting of that part of $\mathfrak{S}$ which is outside S'; then we have

$$\int U\frac{dV}{d\nu}dS + \int U\frac{dV}{d\nu}dS' + \int U\nabla^2 V d\mathfrak{S}'$$

$$= \int V\frac{dU}{d\nu}dS + \int V\frac{dU'}{d\nu}dS' + e\int V\frac{d}{d\nu}\left(\frac{1}{r}\right)dS'$$

$$+ \int V\nabla^2 U' d\mathfrak{S}' + \int V\nabla^2\left(\frac{e}{r}\right)d\mathfrak{S}'$$

$$= -\int\left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\frac{dV}{dz}\right)d\mathfrak{S}'. \qquad (11)$$

If we take P for origin, and if Ω denote the volume of the sphere S', we have

$$d\Omega = r^2\,dr\,d\omega, \quad \text{and} \quad dS' = a^2\,d\omega,$$

where $d\omega$ is the element of a solid angle having its vertex at P; also the value of

$$\frac{d}{d\nu}\left(\frac{1}{r}\right) \text{ at the surface } S' \text{ is } -\frac{1}{a^2}.$$

We have then,

$$\left.\begin{aligned}
&\int U\frac{dV}{d\nu}\,dS' = a^2\int U'\frac{dV}{d\nu}\,d\omega + ae\int\frac{dV}{d\nu}\,d\omega,\\
&\int V\frac{dU'}{d\nu}\,dS' = a^2\int V\frac{dU'}{d\nu}\,d\omega, \quad \int V\frac{d}{d\nu}\left(\frac{1}{r}\right)dS' = -\int V d\omega,\\
&\int U\nabla^2 V d\Omega = \int U'\nabla^2 V d\Omega + e\int\nabla^2 V r\,dr\,d\omega,\\
&\int\frac{dU}{dx}\frac{dV}{dx}\,d\Omega = \int\frac{dU'}{dx}\frac{dV}{dx}\,d\Omega - e\int\frac{dV}{dx}\cos a\,dr\,d\omega,
\end{aligned}\right\}\quad(12)$$

where a is the angle which r makes with the axis of x; also $\nabla^2\left(\frac{e}{r}\right) = 0$ at every point of the field $\mathfrak{S}'$.

If we now suppose a to become infinitely small, we see from equations (12) that

$$\int U\frac{dV}{d\nu}\,dS' = 0, \quad \int V\frac{dU'}{d\nu}\,dS' = 0,$$

$$\int V\frac{d}{d\nu}\left(\frac{1}{r}\right)dS' = -4\pi V_P;$$

and, since $\mathfrak{S} = \mathfrak{S}' + \Omega$, we see also that

$$\int U\nabla^2 V d\mathfrak{S}' = \int U\nabla^2 V d\mathfrak{S},$$

$$\int\frac{dU}{dx}\frac{dV}{dx}\,d\mathfrak{S}' = \int\frac{dU}{dx}\frac{dV}{dx}\,d\mathfrak{S};$$

thus (11) becomes the same as (10).

60. **Uniplanar Form of Green's Theorems.**—When U and V are functions of the coordinates of a point which is limited to the plane of xy, equation (5) assumes the form

$$\int U\frac{dV}{d\nu}\,ds + \iint U\nabla^2 V\,dx\,dy = \int V\frac{dU}{d\nu}\,ds + \iint V\nabla^2 U\,dx\,dy$$

$$= -\iint\left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy}\right)dx\,dy. \qquad (13)$$

If $U = U' + e \log \frac{1}{r}$, where r is the distance of any point from a point P inside the field, since $r \log r = 0$ when $r = 0$, we obtain as the uniplanar equation corresponding to (10)

$$\int U \frac{dV}{d\nu} ds + \iint U \nabla^2 V dx dy = \int V \frac{dU}{d\nu} ds - 2\pi e V_P + \iint V \nabla^2 U' dx dy$$

$$= -\iint \left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} \right) dx\, dy, \qquad (14)$$

where V_P is the value of V at the point P.

61. **Constant Potential.**—If a closed equipotential surface have no mass inside it, the potential is constant for the whole of the internal space.

For, since V is constant over the equipotential surface S, and there is no mass inside it, by Art. 26 we have

$$\int V \frac{dV}{d\nu} dS = 0,$$

also $\nabla^2 V = 0$ at every point inside S; hence, by (9),

$$\iint \left\{ \left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right\} d\mathfrak{S} = 0;$$

and since every term here is positive, we must have

$$\frac{dV}{dx} = \frac{dV}{dy} = \frac{dV}{dz} = 0$$

throughout the whole space inside S, whence V must have the same value throughout this space as at the boundary.

A similar theorem holds good for an uniplanar distribution of mass acting inversely as the distance.

Again, if the potential have a constant value C for any finite portion A of unoccupied space, it must have the same value for the whole of space which can be reached from A without passing through mass.

For if in any portion B of space adjacent to A the potential be everywhere greater than C, we can describe a sphere,

of radius R, having its centre and part of its surface S in A and the rest of its surface in B; then $\int VdS$ taken over the surface S is greater than $\int CdS$, that is, than $4\pi R^2 C$; but by (1), Art. 56, we have $\int VdS = 4\pi R^2 C$. Hence there cannot be a region B adjacent to A throughout which $V > C$. In like manner we can show that there cannot be an adjacent region throughout which $V < C$; but since V is continuous, if it be not equal to C everywhere in the space adjacent to A, there must be some finite portion of this space throughout which either $V > C$, or else $V < C$, either of which alternatives is impossible.

Thus A can be continually increased by the addition of adjacent portions of space so long as no mass is encountered.

In the case of a *uniplanar* distribution of mass acting inversely as the distance, we can, by means of Art. 57, show in like manner that, if the potential be constant throughout a finite unoccupied area A, it has the same constant value for the whole of the plane which can be reached from A without passing through mass.

When a mass distribution is symmetrical round a straight line, if the potential V have a constant value C for a finite portion l of this line situated in unoccupied space, it must have the same constant value for the whole of space which can be reached from l without passing through mass.

To prove this, draw a plane through the axis of symmetry, and if V be not equal to C in the part of the plane adjacent to l there must be a finite portion B of this plane, having l for a boundary, throughout which V is everywhere greater or everywhere less than C; and if the plane be made to revolve round the axis, this must be true for the region generated by the revolution of the area B. If now a sphere be described in this region, having its centre in l, we see, as before, that the supposed inequality of V is impossible.

62. **Potential Zero.**—If the potential be zero at each point of a closed surface, outside which there is no mass, it is zero for the whole of external space, and the total mass is also zero. Conversely, if the total mass be zero, and the potential have a constant value C for a closed surface S

surrounding the entire mass, C is zero, and the potential is zero for the whole of space external to S.

These two theorems are an immediate consequence of equation (9). The boundaries of the region $\mathfrak{S}$ are in this case the closed surface S, and a sphere S' at infinity. If R be the radius of this sphere, and M the total mass, we have

$$\int V\frac{dV}{d\nu}\,dS' = \int \frac{M}{R}\,\frac{M}{R^2}\,R^2\,d\omega = 0, \text{ since } R \text{ is infinite.}$$

Hence, as $\nabla^2 V = 0$ throughout $\mathfrak{S}$, we get

$$-\int\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\} d\mathfrak{S} = \int V\frac{dV}{d\nu}\,dS = 0,$$

when V is zero on S; whence, as in Art. 61, V has the same value throughout $\mathfrak{S}$ as on the surface S, and is therefore zero. Hence again at each point of any closed surface surrounding S we have $\dfrac{dV}{d\nu} = 0$, and therefore, by Art. 26, the total mass is zero.

Again, if V be constant over S, and the total mass zero,

$$\int V\frac{dV}{d\nu}\,dS = 0,$$

and therefore, as before, V is constant throughout $\mathfrak{S}$, and is therefore zero since it is zero at infinity.

Another theorem, which is more general than those given above, and which may be proved in a similar manner, may be enunciated thus:—If the potential be zero at every point of the boundaries of a region in which there is no mass, the potential is zero throughout the region.

63. **Zero Potential for Uniplanar Distribution.**— If at a point P the potential of a uniplanar distribution of mass acting inversely as the distance be zero independently of the absolute magnitude of the unit of length, the total mass must be zero.

For the expression for the uniplanar potential V at a point P is $\Sigma m \log \frac{L}{r}$, where L is the unit of length, and r the distance of the mass m from P, the lengths r and L being expressed in terms of the same unit. If we choose another absolute length L' instead of L as the unit, the expression for V becomes $\kappa \Sigma m \log \frac{L'}{r}$, where κ is a constant depending on L and L'. If this be zero, $\Sigma m \log \frac{L'}{r} = 0$; and if the first expression for V be also zero we have, by subtraction,

$$(\Sigma m) \log \frac{L'}{L} = 0,$$

and therefore $\Sigma m = 0$.

It is now easy to see that, if the uniplanar potential be zero, independently of the absolute magnitude of the unit of length, at every point of a closed curve s, outside which there is no mass, the potential is zero for the whole of the plane outside s, and the total mass is also zero; and conversely, that if the total mass be zero, and the potential have a constant value C for a closed curve s surrounding the entire mass, C is zero, and the potential is zero for the whole of the plane outside s.

In fact, from what has been said, it appears that in each of the supposed cases the total mass is zero, and therefore, by Art. 44, the potential at infinity is zero, and as $\frac{dV}{d\nu}\,ds$ at a point on a circle at infinity is $(\Sigma m)\,d\theta$, if we integrate round this circle, we find that $\int V \frac{dV}{d\nu}\,ds$ is zero. The theorems above are proved then in the same manner as the corresponding theorems for a three-dimensional distribution of mass.

64. **Equivalent Distributions.**—If two distributions, A and B, of mass have the same potential at every point of the boundary of a region to which both distributions are external, they have the same potential at every point throughout the region.

To prove this, suppose one distribution A to be reversed, that is, suppose the algebraical sign of each element of mass reversed, and suppose the distribution thus obtained to coexist with the other. The potential V due to this conjoint distribution is then zero over the boundary of the field which is unoccupied, and therefore, by Art. 62, V is zero throughout the field.

Hence, at every point the potential due to A must be equal to that due to B.

It is to be observed that the boundary of the region here considered may consist of any number of separate surfaces.

As a particular case of the theorem above, we have the important result, that—

If two distributions of mass have the same potential at every point of a closed surface enclosing both, they have the same potential throughout the whole of external space.

Again, it readily appears from Art. 61, that—

If two distributions of mass give potentials which have a *constant difference* C at every point of a closed surface S to which both distributions are external, the potentials due to these two distributions differ by C at every point of the region enclosed by S.

Another theorem which is easily deduced from Green's equation, Art. 58, is, that—

If two distributions of mass produce the same normal force at every point of the boundary of a region to which they are both external, their potentials throughout this region can differ only by a constant.

For, as before, suppose one distribution when reversed to co-exist with the other, and let V be the potential due to the conjoint distribution, then $\frac{dV}{d\nu}$ is zero at every point of the boundary, and as $\nabla^2 V = 0$ throughout the field, by equation (9) we have V constant throughout the field, that is, the difference of the potentials due to the two distributions is constant.

Examples.

1. Prove that the mean value of the potential, due to any system of mass acting inversely as the square of the distance, throughout a sphere having none of this mass inside it, is equal to the potential at the centre of the sphere.

Suppose the sphere divided into an infinite number of shells concentric with it, then the result given above follows from Art. 56.

2. Prove that the mean value of the potential due to any system of uniplanar mass, acting inversely as the distance, throughout a circle having none of this mass inside it is equal to the value of the potential at the centre of the circle.

3. If two distributions, A and B, of mass have the same closed surface S enclosing both as an equipotential, every surface outside S which is an equipotential for A is also an equipotential for B.

Let V_1 be the potential due to A, and V_2 that due to B, then

$$V_1 = C_1 \quad \text{and} \quad V_2 = C_2 \quad \text{on the surface } S.$$

Now imagine a distribution A' whose elements of mass occupy the same positions as those of A, but such that any mass m' belonging to this distribution is given by the equation $m' = \frac{C_2}{C_1} m$, where m is the corresponding mass belonging to A. Then, if V'_1 be the potential due to A', we have $V'_1 = V_2$ at the surface S, and therefore, $V'_1 = V_2$, and $V_1 = \frac{C_1}{C_2} V_2$ for all space external to S.

Hence the equipotential surfaces are the same for the two distributions A and B, and the resultant forces due to them are codirectional, and in a constant ratio to each other.

4. If two distributions, A and B, of mass, which are both external to a region $\mathfrak{S}$, produce normal forces at each point of the boundary of $\mathfrak{S}$ which have everywhere the same ratio to each other, then throughout the region $\mathfrak{S}$ the resultant force due to A is codirectional with that due to B, and has to it a constant ratio.

5. If two distributions, A and B, of mass produce throughout a region $\mathfrak{S}$, to which they are both external, codirectional resultant forces, the ratio of the magnitudes of these forces is constant.

Throughout $\mathfrak{S}$ the equipotential surfaces and tubes of force due to A coincide with those due to B. Let Σ_1 and Σ_2 be two orthogonal sections of an infinitely thin tube of force T, and P_1 and P_2 the corresponding resultant forces due to A, then $P_1\Sigma_1 = P_2\Sigma_2$.

Again, if Q_1 and Q_2 be the resultant forces due to B at the same points, $Q_1\Sigma_1 = Q_2\Sigma_2$, and therefore $\frac{P_1}{Q_1} = \frac{P_2}{Q_2}$. Hence along the tube T the resultant forces due respectively to the two distributions are in a constant ratio. If now we consider a circuit composed of infinitely short elements, ds and ds', of two lines of force comprised between two consecutive equipotential surfaces, and of the lines on these surfaces joining the extremities of ds and ds', we have, by the

principle of the Conservation of Energy, $Pds = P'ds'$, and also, $Qds = Q'ds'$, whence, in going from one line of force to another, $\frac{P}{Q} = \frac{P'}{Q'}$. Hence, throughout $\mathfrak{S}$, the ratio $\frac{P}{Q}$ is constant.

6. If two distributions, A and B, of mass have throughout a region $\mathfrak{S}$ to which they are both external the same equipotential surfaces, the resultant forces due to these distributions are throughout the region $\mathfrak{S}$ codirectional and in a constant ratio.

Throughout $\mathfrak{S}$, if U and V be the potentials due to A and B, we have $U = f(V)$ and also $\nabla^2 U = 0$, $\nabla^2 V = 0$. Hence $U = \kappa V + c$ where κ and c are constants.

7. If two distributions, A and B, of uniplanar mass, acting with a force varying inversely as the distance, have the same closed curve s surrounding both as an equipotential, every curve outside s which is an equipotential for A is also an equipotential for B.

Let M_1 be the total mass, and V_1 the potential corresponding to A, and M_2 and V_2 those corresponding to B, and let the values of V_1 and V_2 at s be C_1 and C_2; then, putting $-\frac{M_2}{M_1} = \kappa$, we have $\kappa M_1 + M_2 = 0$.

Now imagine a distribution A' whose elements occupy the same positions as those of A, but such that any mass m' belonging to it is given by the equation $m' = \kappa m$, when m is the corresponding mass belonging to A.

Let the distribution A' coexist with B, and let V be the corresponding potential; then at s we have $V = \kappa C_1 + C_2$; and since V is constant and the total mass zero, by Art 36, we get

$$\int V \frac{dV}{d\nu} ds = 0,$$

when the integral is taken round the curve s. But by Art 44, the integral

$$\int V \frac{dV}{d\nu} ds$$

taken round a circle at infinity is zero; therefore, by (13), Art. 60, V is constant throughout the whole plane outside s, and being zero at infinity is therefore zero. Hence since $\kappa V_1 + V_2 = V = 0$, the distributions A and B have the same equipotential surfaces everywhere in the plane outside s.

65. **Potential a Maximum or a Minimum.**—If the potential V of any system of mass be a maximum at a point P, there is positive mass at P, and if the potential be a minimum, there is negative mass.

To prove this, describe a small sphere S round P as centre, and take P for origin; then if V be a maximum at P, at each point of the surface of this sphere $\frac{dV}{dr}$ is negative,

and therefore so also is $\int \frac{dV}{dr} dS$ taken over the surface; but $\int \frac{dV}{dr} dS = -4\pi M$, where M is the mass inside the sphere; hence M is positive; and as the sphere may be diminished without limit, there must be positive mass at P. If V be a minimum, it can be shown in a similar manner that there is negative mass at P.

It follows from what has been proved above that V *cannot be a maximum or a minimum in unoccupied space.* The same theorem can be proved in a similar manner for a uniplanar distribution of mass acting inversely as the distance.

66. **Variation of the Potential in Space unoccupied by Mass.**—The potential of masses which are outside a closed surface S has at all points inside this surface a value which lies between the extreme values on the surface.

For, let A be the greatest and B the least value of the potential on the surface S; then if anywhere inside the surface V be greater than A, or less than B, there must be a point where it is a maximum or a minimum, which is impossible.

It follows, as a corollary, that if V be constant over S, it is constant throughout its interior.

Similar results hold good for uniplanar mass.

If the potential V of masses inside a closed surface S has at all points of S the same algebraical sign, it has the same sign at all points outside S, and its greatest magnitude irrespective of sign in external space is less than its greatest magnitude on S.

For, if V were positive at every point of S, and negative at any point outside S, since V is zero at infinity, there must be a point outside S at which V is a minimum, but this is impossible. In like manner it can be shown that if V be negative at every point of S it cannot be positive at any point in external space. Again, if V have at any point outside S a value of the same sign, but greater in magnitude than any of its values on S, it must be a maximum or a minimum at some point in external space which is impossible. Lastly, if A be the greatest value of V irrespective of sign

at the surface S, the value of V at an external point P cannot be equal to A except A be zero. For, if V were equal to A at P, describe a sphere round P as centre not meeting or enclosing S; then by Art. 56, since V has the same algebraical sign at all points of the surface of this sphere, and nowhere exceeds A in magnitude, it must be equal to A at all points of this surface, and therefore for all the included region, and therefore for all space external to S. Hence A must be zero.

By describing a sphere enclosing the surface S, it is now easy to see, by Art. 56, that—

If the potential of masses inside a closed surface S have at every point of this surface the same algebraical sign, the sum of the masses has the same sign as the potential at S.

As an immediate consequence from the above it follows that—

The potential of any system of masses whose sum is zero cannot have the same algebraical sign at every point of a closed surface enclosing these masses.

67. **Points and Lines of Equilibrium.**—A point of equilibrium is one at which the resultant force is zero. At such a point, if V be the potential of the acting mass,

$$\frac{dV}{dx} = 0, \quad \frac{dV}{dy} = 0, \quad \frac{dV}{dz} = 0.$$

A point of equilibrium is therefore a double point on the equipotential surface $V - K = 0$.

If at every point of a line of any form $\frac{dV}{dx}$, $\frac{dV}{dy}$, and $\frac{dV}{dz}$ be zero, the line is a line of equilibrium, and must be a double line on an equipotential surface, which accordingly must have at least two sheets intersecting in this line.

If an equipotential surface have two sheets cutting one another in unoccupied space, they must intersect at right angles.

To prove this, take a point O on the line of intersection as origin; then, at any near point x, y, z, we have, by Taylor's Theorem,

$$V = V_0 + \left(\frac{dV}{dx}\right)_0 x + \left(\frac{dV}{dy}\right)_0 y + \left(\frac{dV}{dz}\right)_0 z + \frac{1}{2}\left\{\left(\frac{d^2V}{dx^2}\right)_0 x^2 + \text{\&c.}\right\} + \text{\&c.},$$

and the equation of the equipotential surface $V - K = 0$, becomes in the neighbourhood of the origin

$$\left(\frac{d^2V}{dx^2}\right)_0 x^2 + \left(\frac{d^2V}{dy^2}\right)_0 y^2 + \left(\frac{d^2V}{dz^2}\right)_0 z^2 + 2\left(\frac{d^2V}{dx\,dy}\right)_0 xy + \&c. = 0;$$

but, since the equipotential surface consists of two sheets intersecting in a line passing through O, the terms of the second degree in the equation above must be of the form

$$(Ax + By + Cz)(A'x + B'y + C'z),$$

where $\quad Ax + By + Cz = 0 \quad$ and $\quad A'x + B'y + C'z = 0$

are the equations of the tangent planes to the two sheets. Hence we have

$$AA' + BB' + CC' = \left(\frac{d^2V}{dx^2}\right)_0 + \left(\frac{d^2V}{dy^2}\right)_0 + \left(\frac{d^2V}{dz^2}\right)_0 = 0,$$

since O is a point in space unoccupied by mass, and therefore the two sheets of the equipotential surface cut at right angles.

68. **Rankine's Theorem.**—If n sheets of an equipotential surface in space unoccupied by mass intersect in the same line, the angle between each pair of adjacent sheets is $\dfrac{\pi}{n}$.

To prove this, take a point O on the multiple line for origin, and a tangent to this line as axis of z; then if O be a multiple point of the order m on the equipotential surface, the tangent cone to this surface at O is of the form $H_m = 0$, where H_m is a homogeneous function of the m^{th} degree in x, y, z. Also, as the tangent planes to the n sheets of the equipotential surface pass through the axis of z, their equation is of the form $u_n = 0$, where u_n is a homogeneous function of the n^{th} degree in x and y; and, since they intersect in a line, on the cone $H_m = 0$, and are tangent planes to this cone, H_m is of the form $u_n z^{m-n} + u_{n+1} z^{m-n-1} + \&c. + u_m$, where u_{n+1}, &c., are homogeneous functions of x and y of the degree $n + 1$, &c. Now in the vicinity of the origin $V = K + H_m + H_{m+1} + \&c.$, where H_{m+1}, &c. are homogeneous functions of x, y, z

of the degree $m+1$, &c.; and, as $\nabla^2 V = 0$ for all values of the coordinates, $\nabla^2 H_m = 0$, and therefore equating to zero the coefficient of the highest power of z in $\nabla^2 H_m$, we get

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) u_n = 0.$$

If p be the perpendicular from any point on the axis of z, we have $u_n = p^n f(\phi)$; and by equation (18), Art. 48, we get

$$\frac{d^2}{dx^2} + \frac{d^2}{dy^2} = \frac{1}{p^2}\left\{\left(p\frac{d}{dp}\right)^2 + \frac{d^2}{d\phi^2}\right\};$$

whence we obtain

$$\frac{d^2 f}{d\phi^2} + n^2 f = 0;$$

and therefore

$$f(\phi) = A \sin(n\phi + a).$$

The value of ϕ for a tangent plane to the equipotential surface is given by the equation $f(\phi) = 0$, and therefore $n\phi + a = s\pi$ when s is an integer.

Hence $\phi_1 = -\frac{a}{n}$, $\phi_2 = \frac{\pi}{n} - \frac{a}{n}$, &c., and $\phi_2 - \phi_1 = \frac{\pi}{n}$, that is, two adjacent sheets of the equipotential surface intersect at the angle $\frac{\pi}{n}$.

In the case of uniplanar mass acting inversely as the distance, it can be shown in a similar manner that if an equipotential curve in a part of the plane unoccupied by mass have a multiple point of the order n, the angle between two adjacent tangents at the point is $\frac{\pi}{n}$.

69. **Diagrams of Equipotential Surfaces.**—If the value of the potential at a point P be known for each of the portions of mass of which a system is composed, we can find the potential of the whole system by addition. This principle enables us to construct the equipotential curves due to a set of centres of force of given intensity. If the centres of force be situated on the same straight line L the equi-

potential surfaces are surfaces of revolution round L as axis. If there be two centres of force, A and B, for which tho corresponding potentials are u and v, take any plane passing through L, and draw in it the circles round A as centre for which $u = 1$, $u = 2$, &c. Draw the equipotential circles also for B; then, if the circles for which $u = m$ and $v = n$ intersect, the value of the total potential V at their point of intersection is $m + n$. Now by altering the values of m and n, their sum remaining constant, we obtain a number of points on the equipotential curve for which $V = m + n$, and thus the curve can be graphically constructed. Having drawn the equipotential curves due to A and B conjointly, we may suppose a third centre of force at a point C on the line joining A and B, and having drawn the equipotential circles corresponding to it we may proceed as before, and so on for any number of centres of force. The method of drawing the lines of force in the case supposed has been already described, Art. 34. By drawing also the equipotential curves in the manner given above, the diagram of the field of force is completed.

70. **Thomson's and Dirichlets' Theorem.**—An acyclic function ϕ of the coordinates always exists, which satisfies the equation $\nabla^2\phi = 0$ throughout a given region $\mathfrak{S}$, and is equal to a given function at each of the boundaries of this region.

There is only one such function.

To prove this, let u be a function of the coordinates satisfying the equation $u = f_1(xyz)$ at the first boundary, $u = f_2(xyz)$ at the second boundary, and so on, and let

$$Q_u = \iiint \left\{ \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 \right\} d\mathfrak{S},$$

taken throughout the region $\mathfrak{S}$; then if v bo another function of the coordinates, and a a constant, we have

$$Q_{u+av} = Q_u + a^2 Q_v + 2a \iiint \left(\frac{du}{dx}\frac{dv}{dx} + \frac{du}{dy}\frac{dv}{dy} + \frac{du}{dz}\frac{dv}{dz} \right) d\mathfrak{S}$$

$$= Q_u + a^2 Q_v - 2a \left\{ \iint v \frac{du}{d\nu} dS + \iiint v \nabla^2 u d\mathfrak{S} \right\}.$$

If v be selected in such a manner as to be zero at each of the boundaries, and so as to make $\iiint v\,\nabla^2 u\, d\mathfrak{S}$ positive, and if a be positive and so small that its square is negligible compared with its first power, we have $Q_{u+av} < Q_u$. Hence, if $u' = u + av$, the function u' satisfies the same boundary conditions as u, but $Q_{u'} < Q_u$. From this it follows that if a function u satisfy the boundary conditions, we can always find another function u' satisfying the same conditions, and such that $Q_{u'} < Q_u$, unless $\nabla^2 u = 0$ throughout the field of integration. Now Q_u is essentially positive and cannot therefore be diminished without limit. Hence a function ϕ exists which satisfies the boundary conditions, and is such that Q_ϕ cannot be made less, but if this be so we must have $\nabla^2\phi = 0$ throughout the field $\mathfrak{S}$.

Again there is only one such function. For if there were two, ϕ and ϕ', and if we put $\phi - \phi' = \chi$, we should have $\chi = 0$ on the boundary, and $\nabla^2\chi = 0$ throughout the field, and therefore have χ zero. Hence $\phi = \phi'$.

A similar theorem holds good for a plane.

It is to be observed that the validity of the proof given above for the first part of Thomson's theorem is not admitted by Weierstrass and other eminent mathematicians.

71. **Surface Distribution.**—It is always possible to distribute mass over a closed surface S so as to produce the same potential in external space as a given distribution of mass M which is inside the surface, or the same potential in internal space as a given distribution M which is outside.

To prove this, let $\mathfrak{S}_1$ be the region outside S, and $\mathfrak{S}_2$ the region inside, ν_1 the normal to S drawn into $\mathfrak{S}_1$, and ν_2 the normal drawn into $\mathfrak{S}_2$; then, if V be the potential due to M, by Art. 70, there is a function ϕ such that $\phi = V$ at S, and is of the order $\dfrac{M}{R}$ at a point P at infinity, where R is the distance of P from the origin, and that $\nabla^2\phi = 0$ throughout $\mathfrak{S}_1$, and a function ψ such that $\psi = V$ at S, and $\nabla^2\psi = 0$ throughout $\mathfrak{S}_2$.

Again, if P be a point in the region $\mathfrak{S}_1$, and r denote the distance of any point from P, by Art. 59, we have

$$\int \frac{1}{r}\frac{d\phi}{d\nu_1}\,dS = \int \phi\,\frac{d}{d\nu_1}\left(\frac{1}{r}\right)dS - 4\pi\phi_P.$$

Also, by Art. 58, we have

$$\int \frac{1}{r}\frac{d\psi}{d\nu_2}\,dS = \int \psi\,\frac{d}{d\nu_2}\left(\frac{1}{r}\right)dS.$$

Adding this equation to the former, and remembering that at the surface S we have $\psi = \phi = V$, and that $\frac{d}{d\nu_1} = -\frac{d}{d\nu_2}$, we get

$$\int \frac{1}{r}\left(\frac{d\phi}{d\nu_1} + \frac{d\psi}{d\nu_2}\right)dS = -4\pi\phi_P. \qquad (15)$$

Hence at any point in $\mathfrak{S}_1$ the function ϕ expresses the potential of a surface distribution whose density is

$$-\frac{1}{4\pi}\left(\frac{d\phi}{d\nu_1} + \frac{d\psi}{d\nu_2}\right).$$

In like manner ψ is the potential in $\mathfrak{S}_2$ of the same surface distribution. Also, if M be situated in $\mathfrak{S}_2$ we have $\phi = V$ throughout $\mathfrak{S}_1$, and if M be situated in $\mathfrak{S}_1$ we have $\psi = V$ throughout $\mathfrak{S}_2$. Hence the surface distribution is determined which produces the same potential *on one side of* S as a given mass distribution existing *on the other side.*

When the mass M is inside S it is equal to the total mass of the surface distribution.

It is obvious that we can show in a similar manner, that if space be divided by a boundary or set of boundaries into two regions, it is always possible to distribute mass over the boundary so as to produce in one region the same potential as that produced by a given distribution of mass existing in the other region. The density of the required surface distribution is determined in the same manner as before.

Another theorem, in some respects more general than those given above, is the following:—

It is always possible to distribute mass over a surface S closed or open so as to produce a potential which is equal at each point of S to a given function of the coordinates.

In this case, if S be an open surface, we have to determine a function ϕ of the coordinates which at a point P at infinity is of the order $\frac{M}{R}$, where M is a finite constant, and R the distance of P from the origin, which equals at S a given function of the coordinates, and which satisfies the equation $\nabla^2\phi = 0$ throughout the whole of space on both sides of S; the density σ of the surface distribution is then determined by the equation

$$4\pi\sigma = -\left(\frac{d\phi}{d\nu_1} + \frac{d\phi}{d\nu_2}\right).$$

72. **Curve Distribution of Uniplanar Mass.**—If V be the potential of a uniplanar distribution of mass M acting inversely as the distance, and s a closed curve surrounding M, a function ϕ of the coordinates x and y exists which is equal to V at the boundary s, and is equal to $M \log \frac{1}{R}$ at a point P at infinity whose distance from the origin is R, and which also satisfies the equation $\nabla^2\phi = 0$ throughout the part of the plane outside s. Proceeding then as in Art. 71, we find that the density υ of a distribution on the curve s producing outside s the same potential as that due to M, is given by the equation

$$2\pi\upsilon = -\left(\frac{d\phi}{d\nu_1} + \frac{d\psi}{d\nu_2}\right).$$

Again, if M be outside s, we find in like manner a distribution on the curve s giving the same potential as M inside s.

The theorems for uniplanar mass corresponding to the remaining theorems of Art. 71 are investigated in like manner.

73. **Distribution on Equipotential Surfaces.**—If the equipotential surfaces corresponding to a distribution of mass M be closed surfaces surrounding M, and if mass be distributed on the equipotential surface S so as to produce in external space the same potential as that produced by M, the density of this distribution at any point of S is inversely proportional to the normal distance at this point between S and the consecutive equipotential surface.

For in this case ψ in equation (15) is constant, and since $\phi = V$, where V is the potential of the mass M, we have

$$V = -\frac{1}{4\pi}\int \frac{dV}{d\nu}\frac{dS}{r}, \tag{16}$$

and σ, the density of the surface distribution producing V in external space, is given by the equation

$$\sigma = -\frac{1}{4\pi}\frac{dV}{d\nu}.$$

But if $V = C$ for the equipotential surface S, at a consecutive equipotential $V' = C + dC$, and

$$\frac{dV}{d\nu}d\nu = V' - V = dC,$$

whence
$$\sigma = -\frac{1}{4\pi}\frac{dC}{d\nu}.$$

It is easy to obtain (16) directly from (10), Art. 59, since at the surface S the potential V is constant, and r being the distance from an external point,

$$\int \frac{d}{d\nu}\left(\frac{1}{r}\right) dS = 0.$$

Similar results hold good for uniplanar mass acting inversely as the distance.

74. **Determination of Equipotentials.**—If S be a closed surface on which there is a distribution of mass whose density at any point is σ, and whose potential is constant on S, and if $d\nu$ be the normal distance between S and a consecutive equipotential surface S'; then, since

$$d\nu = -\frac{1}{4\pi}\frac{dC}{\sigma},$$

if σ be known, the surface S' can be determined.

75. **Distribution of Electricity on a Conductor.**—We have seen, Art. 30, that when a charged conductor is in electric equilibrium, the electric mass is distributed on the

surface so as to produce at each point a resultant force normal to the surface, which is therefore an equipotential surface for this distribution, and the electric mass constitutes what is termed a *couche de niveau*.

If the total amount M of mass be given, there is only one possible distribution consistent with electric equilibrium.

For, if there be two possible distributions, let their potentials in external space be U and V; then at the surface S of the conductor we have $U = C_1$, $V = C_2$; and if ν be the normal to S drawn outwards,

$$4\pi M = -\int \frac{dU}{d\nu}\, dS = -\int \frac{dV}{d\nu}\, dS.$$

Hence

$$\int (U - V)\left(\frac{dU}{d\nu} - \frac{dV}{d\nu}\right) dS = 0;$$

also if $\chi = U - V$, by equation (9), Art. 58, χ is constant throughout the whole of space outside S, and therefore $\frac{d\chi}{d\nu}$ is zero at every point of S; whence the density of the distribution producing U is everywhere the same as that of the distribution producing V.

It can be shown, as in Art. 70, that there is only one possible distribution of mass over a closed surface S producing a given potential at every point of this surface.

Hence, *if an insulated conductor be charged to given potential, the quantity and distribution of electricity on its surface is determined.*

Examples.

1. The potential V is $\phi(x, y, z)$ throughout the region inside a closed surface S, and is zero throughout external space; find the corresponding distribution of mass.

In order that it should be possible to fulfil the required conditions $\phi(x, y, z)$ must be zero at the surface S; then inside S, the density ρ is given by the equation $4\pi\rho + \nabla^2\phi = 0$, and the surface density σ on S by the equation $4\pi\sigma + \frac{d\phi}{d\nu} = 0$, where ν is the normal to S drawn inwards. Since S is an equipotential surface corresponding to V, we have

$$\frac{d\phi}{d\nu} = \pm\sqrt{\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\}}.$$

2. Find the equipotential surfaces of a homogeneous homœoid in external space.

A homœoid is a *couche de niveau*, and therefore the method of Art. 74 is applicable. By Ex. 4, Art, 24, the surface density σ varies as p the central perpendicular on the tangent plane. Now if $a + da$, $b + db$, $c + dc$, be the axes of an ellipsoid confocal with the inner surface of the homœoid, and $p + dp$ the central perpendicular on its tangent plane, since

$$p^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma,$$

we have

$$pdp = ada \cos^2 \alpha + bdb \cos^2 \beta + cdc \cos^2 \gamma;$$

and, as $a^2 - b^2$ and $a^2 - c^2$ are constant, we get $ada = bdb = cdc$, and therefore $pdp = ada$, and p varies as $\frac{1}{dp}$. Hence σ varies as $\frac{1}{dp}$, but, by Art. 74, σ varies as $\frac{1}{d\nu}$, that is, the consecutive equipotential surface of the homœoid is an ellipsoid confocal with its inner surface. A surface distribution on this, giving the same potential as that of the homœoid, is a *couch de niveau*, and (Art. 75) is therefore, another homœoid, whose equipotential is another ellipsoid confocal with the inner surface of the original homœoid. Proceeding in this manner we see that the equipotential surfaces of a homœoid are confocal ellipsoids, and also, that confocal homœoids of equal mass (Art. 37) have the same potential in external space.

A fuller account of the properties of confocal homœoids will be found in Chapter V.

3. Find the potential V of a homœoid H at any point P in external space.

Since the equipotential surfaces of H are confocal ellipsoids, V varies only along with the axis major $2a'$ of the confocal ellipsoid E' passing through P, and therefore $V = f(a')$. Let $2(a' + \delta a')$ be the axis major of the outer surface of a homœoid H' whose inner surface is E', then if M' be its mass, and ρ' its volume density, we have

$$M' = \frac{4}{3} \pi\rho' \ \delta \ (a'b'c') = 4\pi\rho' \, b'c'\delta a',$$

but if H' and H have the same potential in space external to both, $M' = M$, and therefore $4\pi\rho'\delta a' = \frac{M}{b'c'}$. Now if σ' be the surface density of the distribution on the ellipsoid E' which produces V, at the point which is at the extremity of the axis major of E' we have $\sigma' = \rho'\delta a'$, and also, by Arts. 46 and 61, we get

$$-4\pi\sigma' = \frac{dV}{da'}.$$

Hence,

$$\frac{dV}{da'} = -\frac{M}{b'c'} = -\frac{M}{\sqrt{(a'^2 - h^2)(a'^2 - k^2)}}.$$

Integrating, and remembering that $V = 0$ when $a' = \infty$, we get

$$V = M\int_{a'}^{\infty} \frac{da'}{\sqrt{(a'^2 - h^2)(a'^2 - k^2)}},$$

where h and k are the constants of the confocal system. If we transform the integral by assuming $a' \sin\theta = k$, we get

$$V = \frac{M}{k} F\left(\frac{h}{k}, \theta\right),$$

where F is an elliptic integral of the first kind whose modulus is $\frac{h}{k}$, and whose amplitude is θ.

4. Show that the potentials of two confocal homœoids at any point outside both are proportional to their masses.

5. Prove that the potentials of confocal homogeneous ellipsoids, in space outside both, are proportional to their masses.

Let E and E' be the ellipsoids, the semi-axes of E being a, b, c, and those of E' being a', b', c', then if λa, λb, λc be the semiaxes of an ellipsoid similar to E, and $\lambda a'$, $\lambda b'$, $\lambda c'$ those of an ellipsoid similar to E', since, $a^2 - b^2 = a'^2 - b'^2$, and $a^2 - c^2 = a'^2 - c'^2$, we have $\lambda^2(a^2 - b^2) = \lambda^2(a'^2 - b'^2)$, $\lambda^2(a^2 - c^2) = \lambda^2(a'^2 - c'^2)$, and if λ go from 0 to 1, the ellipsoids E and E' are each divided into an infinite number of homœoids, the bounding surfaces of the homœoids composing E being confocal with those of the corresponding homœoids of E'. Also the mass H of any homœoid in E is $4\pi\rho\lambda^2 abcd\lambda$, and the mass H' of the corresponding homœoid in E' is $4\pi\rho'\lambda^2 a'b'c'd\lambda$, and therefore $\frac{H}{H'} = \frac{M}{M'}$, where M and M' are the masses of the ellipsoids. Hence as the potential of E is the sum of the potentials of the homœoids of which it is composed, and a similar statement holds good for E', and as the potentials of each pair of corresponding homœoids in external space are proportional to M and M', this is also true of the potentials of the ellipsoids.

The theorem above is usually ascribed to Mac Claurin.

6. Find the equipotential curves due to an elliptic homœoidal band of uniplanar mass acting inversely as the distance.

By an investigation similar to that of Ex. 2, it can be shown that the equipotential curves required are confocal ellipses.

7. Find the potential of an elliptic homœoidal band of uniplanar mass in external space.

If M be the uniplanar mass, and V the potential at the point P, of the homœoidal band H, a' the semiaxis major of the ellipse passing through P confocal with H, and $2c$ the distance between its foci, as in Ex. 3, we have

$$\frac{dV}{da'} = -\frac{M}{\sqrt{(a'^2 - c^2)}}.$$

Assuming $a' = c\cosh\eta$, and integrating, we get $V = C - M\eta$.

When a' is infinite,

$$V = -M\log a' = -M\{\eta + \log \tfrac{1}{2}c\};$$

and therefore, in general,

$$V = M\{\log 2 - \log c - \eta\}.$$

8. A distribution of mass M has for an equipotential surface an ellipsoid E enclosing M; find the potential of M at any point P outside E.

If V be the potential required,

$$V = M\int_{a'}^{\infty} \frac{da'}{\sqrt{(a'^2 - h^2)(a'^2 - k^2)}},$$

where $2a'$ is the axis major of an ellipsoid passing through P confocal with E, and h and k are the constants of the confocal system.

9. An insulated ellipsoidal conductor is charged with a quantity E of electricity; find its potential at any point P in external space.

If V be the potential,

$$V = E\int_{a'}^{\infty} \frac{da'}{\sqrt{(a'^2 - h^2)(a'^2 - k^2)}},$$

where a', &c., have the same meaning as in Ex. 8.

10. If two distributions of mass, M and M', in portions of space $\mathfrak{S}$ and $\mathfrak{S}'$ separated from each other by a continuous surface S, produce tangential forces equal and in the same direction at every point of S, a surface distribution on S can be effected which produces the same resultant force throughout $\mathfrak{S}$ as that produced by M', and the same throughout $\mathfrak{S}'$ as that produced by M. (Thomson and Tait.)

If V and V' be the potentials due to M and M', and if there be a surface distribution whose potential is U, and such that $U = V$ on S; then by Art. 64 $U = V$ throughout $\mathfrak{S}'$. But at S we have $V - V' = C$; hence $U = V' + C$ at S, and therefore $U = V' + C$ throughout $\mathfrak{S}$ by Art. 64.

11. A hollow conductor, comprised between two closed surfaces, has electric mass in its interior: show that the whole potential in external space is that due to the charge on the external surface S.

Let v be the potential due to the charge on the outer surface, and let V be the total potential; then if r be the distance from any point P in the region $\mathfrak{S}$ outside S, by Art. 59 we have,

$$\int \frac{1}{r}\frac{dV}{d\nu}\,dS + \int \frac{1}{r}\,\nabla^2 V d\mathfrak{S} = \int V \frac{d}{d\nu}\left(\frac{1}{r}\right)dS - 4\pi V_P,$$

whence, as $\nabla^2 V = 0$ throughout $\mathfrak{S}$, and V is constant on S, we get

$$4\pi V_P = -\int \frac{dV}{d\nu}\frac{dS}{r}.$$

Again if σ be the density of the surface distribution on S, by Art. 46 we have

$$4\pi\sigma + \frac{dV}{d\nu} + \frac{dV}{d\nu'} = 0,$$

and since there is no force in the substance of the conductor

$$\frac{dV}{d\nu'} = 0,$$

also
$$v = \int \frac{\sigma dS}{r}.$$

Hence, substituting for σ, we obtain $v_P = V_P$.

12. In Ex. 11, show that the total electric mass on the interior surface of the conductor is equal in magnitude and opposite in algebraical sign to the total mass in the interior hollow, and that the potential of the two distributions conjointly is zero in the substance of the conductor.

If u be the potential due to the mass in the interior hollow in conjunction with that on the inner surface, and V and v have the same meaning as in Ex. 11, then $V = v + u$, but in external space $V = v$, and therefore $u = 0$, and being due to mass none of which is outside the inner surface, u must be zero up to this surface, Art 61; hence by Art. 26 the total mass producing u is zero.

13. A hollow conductor, whose external surface is an ellipsoid, has electric mass E' in its interior, and receives a charge E; find the potential in external space.

If the conductor were uncharged, there would be in consequence of the mass in its interior a charge $-E'$ on its inner surface, and a charge $+E'$ on its outer surface forming a *couche de niveau*. If now a charge E be communicated, the total charge on the outer surface becomes $E+E'$; and as it forms a *couche de niveau*, the potential is obtained by putting $E+E'$ for E in the answer to Ex. 9.

14. A material particle, under the action of forces due to mass attracting or repelling inversely as the square of the distance, and situated in space unoccupied by this mass, cannot be in stable equilibrium.

15. If a particle be placed at a point of equilibrium O in unoccupied space, and receive a small displacement R of given magnitude, show that the force which acts on the particle in the direction of R varies inversely as the square of the codirectional radius vector of a hyperboloid having O as centre.

If the coordinates of O be x, y, z, the potential V at a point in the vicinity of O, whose relative coordinates are ξ, η, ζ, is given by the equation

$$V = V_0 + \tfrac{1}{2}\left(\frac{d^2V}{dx^2}\xi^2 + 2\frac{d^2V}{dxdy}\xi\eta + \text{\&c.}\right);$$

the equipotential surfaces in the vicinity of O are therefore hyperboloids; and if we take their principal axes as axes of coordinates, we have

$$V = V_0 + \tfrac{1}{2}(A\xi^2 + B\eta^2 + C\zeta^2);$$

then the components of the force acting on the particle are $-AR\cos\alpha$,

$-BR\cos\beta$, $-CR\cos\gamma$, where α, β, γ are the direction angles of R; whence the force along R is $-(A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma)\,R$, or $-\dfrac{K}{r^2}R$, where r is the radius vector of the hyperboloid whose equation is $A\xi^2 + B\eta^2 + C\zeta^2 = K$.

16. In the last Example, if p be the central perpendicular on the tangent plane to the hyperboloid at the point where it is met by r, show that the total force acting on the particle is in the direction of p, and is inversely proportional to pr.

It is to be observed that in this and the preceding Example, when the direction of R is such that $A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma$ is negative, we must use the hyperboloid $A\xi^2 + B\eta^2 + C\zeta^2 = -K$.

17. Prove that at O, a point of equilibrium, sets of three mutually perpendicular lines can be found such that a particle at O displaced along one of them is not acted on by any force in the direction of this displacement.

If we suppose the axis of ζ to lie on the cone Γ whose equation referred to its principal axes is $Ax^2 + By^2 + Cz^2 = 0$, since then the coefficient of ζ^2 must vanish, and since the invariant $A + B + C = 0$, the equation of Γ assumes the form

$$a(\xi^2 - \eta^2) + 2h\xi\eta + 2f\eta\zeta + 2g\zeta\xi = 0.$$

But as the factors, X and Y, of $a(\xi^2 - \eta^2) + 2h\xi\eta$ are necessarily real, and represent when equated to zero perpendicular planes, the equation of Γ becomes $XY + L\zeta(MX + NY) = 0$. Hence the lines of intersection of the planes X, Y; X, ζ; and Y, ζ, are mutually perpendicular, and are all edges of the cone Γ. A particle, therefore, displaced from O along one of these lines remains on the equipotential surface passing through O, and is therefore unacted on by any force tangential to this surface.

18. If a distribution of mass M consist of two parts, M_1 and M_2, round each of which a closed surface can be drawn, the intervening space being unoccupied, and if for the whole of space outside these surfaces, S_1 and S_2, the distribution M be centrobaric, prove that the baric centre must be inside either S_1 or S_2, and that if it lie inside S_1, the potential of the mass inside S_2 must be zero for the whole of space outside S_2.

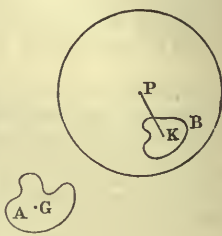
If G, the baric centre, be outside both S_1 and S_2, describe a surface S in the space external to S_1 and S_2 enclosing G; then, if N be the normal force at any point of this surface S, since M is outside S, we have $\int NdS = 0$; and therefore, throughout all space outside S_1 and S_2 the potential is that due to a zero mass placed at G, and is therefore zero. Hence G must lie inside either S_1 or S_2. If G be inside S_1, imagine a surface distribution on S_2 giving the same potential as M_2 through space outside S_2, and let V be the potential due to this distribution coexisting with the distribution M_1 and the mass $-M$ placed at G, then V is zero throughout all space external to S_1 and S_2, and being zero on the surface S_2 is zero throughout the enclosed region. Hence $\dfrac{dV}{d\nu}$ is zero at both sides of S_2, and the surface density at each point of S_2 is zero, and the potential in external space of the supposed distribution on S_2 is zero, and therefore also the potential due to M_2.

19. If a system of mass M be centrobaric throughout a finite portion $\mathfrak{S}$ of space outside M, it is centrobaric for the whole of space outside itself.

Throughout $\mathfrak{S}$ the potential of M is the same as that of a mass M concentrated at a point G. Hence by Art. 61 the potentials of these two distributions are the same throughout the whole of space outside M.

20. If the system of forces exerted by an invariable mass system A on another invariable mass system B be always reducible to a single force passing through the same point G in A whatever be the position of B, then A is centrobaric for all space outside itself.

Take a point P so distant from A that a sphere can be described, with P as centre, not meeting A, and capable of containing B between its centre and its surface; place B in this position, and draw any line PK meeting B. Concentric spheres having P as centre will pass through successive layers of B, and each layer will constitute a figure on the sphere on which it is situated. If B be made to rotate round PK, the set of particles lying on the sphere whose radius is PK generate by their successive positions a spherical figure bounded by a circle of which K is pole. The successive copolar circles of which this area is composed have been passed over by sets of particles which are different for each circle, but each point on the same circle has been passed over by the same number of particles. Hence if we suppose the successive positions of B to coexist, the density of B being changed from ρ to $\frac{\rho}{n}$, where n is infinite, we obtain a solid of revolution B' formed by a series of layers on successive concentric spheres, each layer being bounded by a circle, and such that the density at any point in the layer depends solely on its distance from the pole of the bounding circle.



If we now suppose the line PK with the solid B' rigidly attached to take every possible position round the point P, and all these positions to coexist, the density being again divided by an infinite constant, we obtain on the sphere having PK as radius a homogeneous shell, since the conditions determining the density are perfectly symmetrical for all points on this sphere, and on the whole we obtain a set of concentric spherical shells, each of which is homogeneous. Hence the resultant attraction of A on a set of concentric homogeneous spherical shells having P as centre passes through G; and, therefore, the attraction of A on a mass concentrated at P passes through G.

It can be shown in like manner that this holds for every position of P throughout a finite volume. Hence A is centrobaric for the whole of space outside itself.

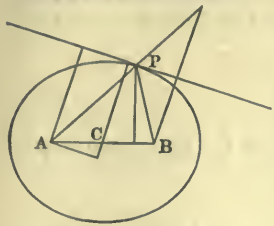
Another method of proving the above theorem will be found in Art. 81, Ex. 8.

21. The surface density on a conductor under the influence of other conductors is positive in some places, and negative in others; show that the equipotential surface for which the potential is that of the conductor has more than one sheet.

22. Find the equipotential surfaces belonging to a homogeneous thin bar AB whose density is λ.

Let P be any point outside the bar, by Art. 10 the resultant force at P is in the direction of the bisector of the angle APB, that is, it is along the normal to an ellipse passing through P and having A and B as foci. Hence the equipotential surfaces required are confocal ellipsoids of revolution.

23. Find the distribution of mass on one of the equipotential surfaces which produces the same potential as that produced by the bar in external space.



Let $AB = 2c$, $APB = \phi$, $AP = r_1$, $BP = r_2$, $r_1 + r_2 = 2a$, let C be the middle point of AB, and let the perpendiculars from A, C, and B, on the external bisector of the angle APB be denoted by p_1, p, p_2, and the perpendicular from P on AB by q. Then, if m be the mass of the bar, and R the resultant force at P, by Art. 10, we have

$$R = \frac{2\lambda \sin \frac{1}{2}\phi}{q} = \frac{2\lambda c \sin \frac{1}{2}\phi}{cq} = \frac{2m \sin \frac{1}{2}\phi}{r_1 r_2 \sin \phi} = \frac{m}{r_1 r_2 \cos \frac{1}{2}\phi}$$

$$= \frac{ma}{r_1 r_2 a \cos \frac{1}{2}\phi} = \frac{ma}{r_1 r_2 p};$$

but, by a well-known property of the ellipse, we have

$$r_1 r_2 = \frac{a^2 b^2}{p^2};$$

whence, by substitution, we get

$$R = \frac{mp}{ab^2} = \frac{mp}{a(a^2 - c^2)};$$

and if σ be the density of the required surface distribution, we have,

$$4\pi\sigma = \frac{mp}{a(a^2 - c^2)},$$

where p is the central perpendicular on the tangent, and a the semiaxis major, of the ellipse passing through P and having A and B as foci.

This result can otherwise be deduced from Ex. 22, by means of Ex. 2 and Ex. 3.

76. **Green's Function.**—If there be a closed surface S, and two points P and Q, on the same side of this surface, Green's function is the potential at Q of a surface distribution on S which, in conjunction with a unit of mass at P, produces a zero potential at all points of S.

Let p denote the density of this distribution; then, if Green's function be denoted by G_{PQ}, we have

$$G_{PQ} = \int \frac{pdS}{AQ},$$

where A is any point on the surface S.

We can now show that if U denote the potential of any distribution on S, the potential U_P of this distribution at P, is given by the equation

$$U_P = -\int p\,U dS. \tag{17}$$

To prove this, let σ be the density of the distribution producing U; then

$$U_P = \int \frac{\sigma dS}{AP}, \text{ but } -\frac{1}{PA} = G_{PA} = \int \frac{p' dS'}{AA'},$$

where A' is any point on S different from A. Substituting we have

$$U_P = -\int \sigma dS \int \frac{p' dS'}{AA'} = -\int p' dS' \int \frac{\sigma dS}{AA'} = -\int U_{A'}\, p' dS',$$

which is the same as the right-hand member of (17).

It follows readily from (17) that no alteration is produced in Green's function by interchanging P and Q. For, let $U_Q = G_{PQ}$, then, if q be the density of a surface distribution corresponding to a unit mass at Q, we have

$$G_{PQ} = -\int G_{PA}\, q dS = -\iint \frac{p' q dS dS'}{AA'} = G_{QP}.$$

77. Energy in Terms of Resultant Force.—The potential energy due to the mutual action of mass which is continuously distributed through a volume or over a surface may be expressed in terms of the resultant force.

If V denote the potential due to the mass, S the surface or surfaces on which there is a distribution, $\mathfrak{S}$ space on both sides of these surfaces, m the mass concentrated at any point, R the resultant force, and W the energy required to produce the distribution; by (21), Art. 50, we have $W = \frac{1}{2}\Sigma m V$; but if σ be the density on a surface where there is a distribution, by Art. 46, we have

$$m = \sigma dS = -\frac{1}{4\pi}\left(\frac{dV}{d\nu} + \frac{dV}{d\nu'}\right) dS;$$

also throughout the field on both sides of this surface

$$m = \rho d\mathfrak{S} = -\frac{1}{4\pi}\nabla^2 V d\mathfrak{S}.$$

Hence, by (9), Art. 58, we obtain

$$8\pi W = -\int V\left(\frac{dV}{d\nu} + \frac{dV}{d\nu'}\right) dS - \int V\nabla^2 V d\mathfrak{S} = \int R^2 d\mathfrak{S}, \quad (18)$$

where the volume integrals are taken through the whole of space.

Examples.

1. Prove that if the potential be given at every point on a set of surfaces, the potential energy due to the mutual action of the mass producing this potential is least when all the mass is on these surfaces.

Let S denote the set of surfaces, $\mathfrak{S}$ the whole of space on both sides of these surfaces, V the potential of a mass distribution on them such that V has the assigned value everywhere on S, and $V+v$ the potential of any other distribution fulfilling this condition. Then, if Q_v have the same meaning as in Art. 70, by Art. 77 we have

$$8\pi W_{V+v} = Q_{V+v} = Q_V + Q_v + 2\int\left(\frac{dV}{dx}\frac{dv}{dx} + \frac{dV}{dy}\frac{dv}{dy} + \frac{dV}{dz}\frac{dv}{dz}\right)d\mathfrak{S}$$

$$= Q_V + Q_v - 2\int v\left(\frac{dV}{d\nu} + \frac{dV}{d\nu'}\right) dS - 2\int v\nabla^2 V d\mathfrak{S};$$

but v must be zero everywhere on S, and $\nabla^2 V = 0$ throughout $\mathfrak{S}$, since corresponding to V there is no volume distribution; therefore, $W_{V+v} = W_V + W_v$, and therefore the energy corresponding to the distribution producing V is least.

2. Show that of all surface distributions of given charges on a given system of conductors that which is consistent with equilibrium has the least potential energy.

This follows immediately from the general dynamical theorem that in a moveable system the potential energy is least when the system is in stable equilibrium.

It may be proved directly as follows:—

Let S_1, S_2, &c., be the surfaces of the conductors, V the potential due to a surface distribution producing a constant potential on each conductor, and such that each conductor has the assigned charge, $V+v$ that due to any other surface distribution fulfilling the latter condition. Then, if E_1 be the charge on S_1, we have

$$4\pi E_1 = -\int\left(\frac{dV}{d\nu_1} + \frac{dV}{d\nu_1'}\right) dS_1,$$

and also

$$4\pi E_1 = -\int\left(\frac{d(V+v)}{d\nu_1} + \frac{d(V+v)}{d\nu_1'}\right) dS_1;$$

whence

$$\int \left(\frac{dv}{d\nu_1} + \frac{dv}{d\nu_1'}\right) \, dS_1 = 0,$$

and a similar equation holds good for each of the other conductors. Again

$$8\pi W_{V+v} = Q_{V+v} = Q_V + Q_v + 2\int \left(\frac{dV}{dx}\frac{dv}{dx} + \frac{dV}{dy}\frac{dv}{dy} + \frac{dV}{dz}\frac{dv}{dz}\right) d\mathfrak{S}$$

$$= Q_V + Q_v - 2\int V\left(\frac{dv}{d\nu_1} + \frac{dv}{d\nu_1'}\right) dS_1 - 2\int V\left(\frac{dv}{d\nu_2} + \frac{dv}{d\nu_2'}\right) dS_2 - \&c.$$

$$-\int V\nabla^2 v d\mathfrak{S} = Q_V + Q_v,$$

since, by what is said above, the surface integrals at S_1, S_2, &c. vanish, V being constant at each of these surfaces, and, as there is no volume distribution, $\Delta^2 v$ is zero throughout $\mathfrak{S}$. Hence $W_{V+v} = W_V + W_v$, and the energy corresponding to V is the least possible.

3. Prove that a charged body cannot be in stable equilibrium under the action of electric forces.

Let e be the charge in any small portion of the charged body A, and V the potential of the electric forces; then W, the potential energy due to the presence of A in the electric field, would be ΣVe if all the electric mass were rigidly fixed in the bodies in which it is distributed. Let ξ, η, ζ be the coordinates of a point P in the body A, and a, b, c the coordinates of any point Q in A relative to P; the absolute coordinates x, y, z of Q are given then by the equations $x = \xi + a$, $y = \eta + b$, $z = \zeta + c$. For a motion of translation of A, the coordinates a, b, c are constant, and for variations due to such a motion $\dfrac{d}{dx} = \dfrac{d}{d\xi}$, &c. Take a point O in the vicinity of P as origin, and describe a sphere S round O as centre, so small that, when P moves about in this sphere, no part of A can enter the region occupied by the electric masses producing V. Let r be the distance of P from O, let $M = \Sigma Ve$, and suppose P to move about in the region $\mathfrak{S}$ inside S, the corresponding motion of A being one of translation, and all the electric mass being supposed to be rigidly attached to the bodies to which it belongs. Then

$$\int \frac{dM}{dr} dS = \int \left(\frac{d^2}{d\xi^2} + \frac{d^2}{d\eta^2} + \frac{d^2}{d\zeta^2}\right) M d\mathfrak{S} = \int \Sigma e \nabla^2 V d\mathfrak{S} = 0.$$

Hence $\dfrac{dM}{dr}$ cannot have the same sign at all points of S. If we now suppose O to approach infinitely near P, and the sphere to become infinitely small, we see that there must be a displacement δs of P for which $\dfrac{dM}{ds}$ is negative. In the actual state of things the electric mass is free to move on the conductors to which it belongs; and when P receives a displacement δs, the change of the potential energy W is given by the equation $\delta W = \dfrac{dM}{ds}\delta s - K\delta s$, where K is essentially positive. Hence, when $\dfrac{dM}{ds}$ is negative, δW is negative; and therefore it is possible to give a displacement by which the potential energy is diminished, and therefore A cannot be in stable equilibrium.

SECTION III.—*Expansion in Series.*

78. **Potential at Distant Point.**—If the origin O be taken in the interior of a system of attracting or repelling mass, the potential at a point P, which is more distant than any point in this mass from O, may be expressed in a convergent series of descending powers of r, the distance of P from O.

Let x, y, z be the coordinates of P, x', y', z' those of any point in the acting mass, r' its distance from O, dm the element of mass there concentrated, V the potential at P, and M the total mass; then

$$V = \int dm\ \{r^2 + r'^2 - 2(xx' + yy' + zz')\}^{-\frac{1}{2}}$$

$$= \int \frac{dm}{r}\left\{1 - \frac{2(xx' + yy' + zz')}{r^2} + \frac{r'^2}{r^2}\right\}^{-\frac{1}{2}}$$

$$= \int \frac{dm}{r}\left\{1 + \frac{xx' + yy' + zz'}{r^2} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\right.$$

$$\left.\frac{4(xx' + yy' + zz')^2}{r^4} - \frac{1}{2}\frac{r'^2}{r^2} + \&\text{c.}\right\}$$

If we now suppose O to be the centre of inertia of M, and the axes of coordinates to be the principal axes of M at O, we have

$$\int x'dm = \int y'dm = \int z'dm = 0,$$

$$\int x'y'dm = \int y'z'dm = \int z'x'dm = 0,$$

whence

$$V = \frac{M}{r} + \frac{3}{2r^5}\int (x^2x'^2 + y^2y'^2 + z^2z'^2)\ dm - \frac{1}{2r^3}\int r'^2dm + \&\text{c.} \quad (1)$$

If P be so distant that $\left(\frac{r'}{r}\right)^3$ is negligible, it is unnecessary to consider any terms in the series for V except those given above. Let A, B, C denote the principal moments of inertia of M at O, and I its moment of inertia round OP, and let

the direction angles of this line be denoted by α, β, γ, then we have

$$\frac{x}{r} = \cos\alpha,\ \frac{y}{r} = \cos\beta,\ \frac{z}{r} = \cos\gamma,$$

also $\int x'^2 dm = \int r'^2 dm - A$, with two similar equations; and neglecting terms of an order higher than $\left(\frac{r'}{r}\right)^2$, we get

$$V = \frac{M}{r} + \frac{1}{2r^3}\int \{3\,(x'^2\cos^2\alpha + y'^2\cos^2\beta + z'^2\cos^2\gamma) - r'^2\}\,dm$$

$$= \frac{M}{r} + \frac{1}{2r^3}\{2\int r'^2 dm - 3(A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma)\}$$

$$= \frac{M}{r} + \frac{1}{2r^3}(A + B + C - 3I). \tag{2}$$

It follows from equation (1) that, at a point P so distant from M that $\left(\frac{r'}{r}\right)^2$ is negligible, the potential of M is the same as if the entire mass were concentrated at its centre of inertia.

The term *moment of inertia* when applied to electric mass signifies merely an integral depending on the positions and intensities of the force-centres of which the electric mass is composed. A remark of a similar character applies to the term *centre of inertia.*

79. **Moment exerted by Distant Body.**—If M be a rigid body, and a mass L be concentrated at a distant point P, remembering that in the case of mutually attracting bodies the potential is a force function, we see that the moments round the axes of the force which M exerts on L are

$$L\left(x\frac{dV}{dy} - y\frac{dV}{dx}\right), \text{ \&c.};$$

but since these are equal and opposite to the moments exerted by L on M, if these latter be denoted by N_1, N_2, N_3, we have

$$N_3 = L\left(y\frac{dV}{dx} - x\frac{dV}{dy}\right), \text{ \&c.}$$

Substituting for V from (1), neglecting terms which are omitted in (2), and remembering that

$$\left(y\frac{d}{dx} - x\frac{d}{dy}\right)r = 0,$$

we get,

$$N_3 = \frac{3L}{2r^5}xy\,2\int(x'^2 - y'^2)\,dm = \frac{3L}{r^5}(B - A)\,xy.$$

We have then for the three moments required

$$\left.\begin{aligned} N_1 &= \frac{3L}{r^5}(C - B)\,yz, \\ N_2 &= \frac{3L}{r^5}(A - C)\,zx, \\ N_3 &= \frac{3L}{r^5}(B - A)\,xy \end{aligned}\right\}. \qquad (3)$$

80. **Ellipsoid of Small Ellipticity.**—In the case of a homogeneous ellipsoid of small ellipticity, equations (2) and (3) are approximately true, no matter how near the point P is to the surface of the ellipsoid.

To prove this, let a, b, c be the semi-axes of the ellipsoid whose mass is M, and a', b', c' those of a confocal ellipsoid inside M whose mass is M', and moments of inertia A', B', C'; then if V' be the potential of the latter ellipsoid at P, by Ex. 5, Art. 75, we have $V = \frac{M}{M'}V'$.

The largest value of r' for the mass M' is a', and if P be outside M the smallest possible value of r is c. Hence, taking M' as the acting mass, we have $\frac{r'^2}{r^2} < \frac{a'^2}{c^2}$. Now, if the ellipticity $\frac{a-c}{c}$ of the ellipsoid M be denoted by ϵ, on the hypothesis that ϵ^2 is negligible $a^2 - c^2 = 2\epsilon c^2$, whence $a'^2 = c'^2 + 2\epsilon c^2$, and $\frac{a'^2}{c^2} = \frac{c'^2}{c^2} + 2\epsilon$; but c' may be made as small as we please, and therefore $\frac{r'^2}{r^2} < 2\epsilon$, and $\left(\frac{r'}{r}\right)^3$ is negligible, so that

equation (2) holds good for V'. Again,

$$A' + B' + C' - 3I' = A' (\cos^2 \beta + \cos^2 \gamma - 2 \cos^2 \alpha) + \&c.$$
$$= \cos^2 \alpha (B' - A' + C' - A') + \&c.;$$

and as

$$B' - A' = \frac{M'}{5} (a'^2 - b'^2) = \frac{M'}{M} (B - A), \&c.,$$

we have

$$A' + B' + C' - 3I' = \frac{M'}{M} (A + B + C - 3I).$$

Hence equation (2) holds good for V, provided ϵ be so small that its powers higher than the first may be neglected.

Equations (3) being deduced from (2) also hold good for an ellipsoid of small ellipticity no matter how near be the point P to its surface.

It is plain that the above results can be extended to a mass composed of homogeneous shells bounded by coaxal ellipsoids of small ellipticity, since the potential, mass, and moments of inertia of a shell are the differences of the corresponding quantities for the ellipsoids between which the shell is comprised. The results which have been obtained may therefore be regarded as valid in the case of the Earth.

The Theorems of this Article are due to Laplace, but the mode of proof here adopted is that of Mac Cullagh.

81. **Clairaut's Theorem.**—If the Earth be supposed to be formed of homogeneous strata bounded by concentric ellipsoids of revolution of small ellipticity having a common axis, a relation of much importance exists between γ the ratio of the excess of polar over equatorial gravity to the latter of these quantities, ϵ the ellipticity of the external surface of the Earth, and q the ratio of the centrifugal force at the equator to gravity.

This relation which was discovered by Clairaut is expressed by the equation

$$\gamma + \epsilon = \frac{5}{2} q. \qquad (4)$$

The following proof of this proposition is that of Mac Cullagh, but is here presented in an improved form due to Dr. Williamson.

If C denote the moment of inertia of the Earth round its polar axis, and θ the angle which the radius vector r makes with this axis, since in the present case $A = B$, we have

$$A + B + C - 3I = (C - A)(3 \sin^2 \theta - 2);$$

whence

$$V = \frac{M}{r} + \frac{C - A}{2r^3} (3 \sin^2 \theta - 2). \qquad (5)$$

If $2a$ and $2c$ be the equatorial and polar axes of the Earth, the equation of a meridian is

$$r^2 \left(\frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{c^2} \right) = 1, \quad \text{and} \quad \varepsilon = \frac{a - c}{c},$$

whence, if ε^2 be neglected, $r = c\,(1 + \varepsilon \sin^2 \theta)$.

The greater part of the surface of the Earth is covered with liquid in equilibrium, and therefore at the surface; if ω denote the Earth's angular velocity, we have

$$V + \frac{\omega^2 r^2 \sin^2 \theta}{2} = \text{constant};$$

accordingly, after expressing r in terms of θ, we may equate to zero the coefficient of $\sin^2 \theta$ on the left-hand side of this equation; thus, neglecting small quantities of the second order, we get

$$-\frac{M}{c}\varepsilon + \frac{3\,(C - A)}{2c^3} + \frac{\omega^2 c^2}{2} = 0;$$

whence, as $q = \dfrac{\omega^2 a}{g_e} = \dfrac{\omega^2 c^3}{M}$, approximately, we have

$$\frac{3}{2} \frac{C - A}{Mc^2} = \varepsilon - \frac{q}{2}. \qquad (6)$$

The angle which the vertical line or normal at any point of the Earth's surface makes with r is of the order ε, and its cosine differs from unity by a quantity of the order ε^2; also the component of the Earth's attraction perpendicular to r is of the order ε, and this resolved along the normal is of the order ε^2; hence, ε^2 being neglected, the component of the Earth's attraction along the normal is $\dfrac{dV}{dr}$.

Again, the component along r of the centrifugal force is $\omega^2 r \sin^2\theta$, and this may be taken as the component of the same force along the normal. Hence, if g denote the acceleration of gravity at any point, we have

$$g = -\frac{dV}{dr} - \omega^2 r \sin^2\theta.$$

If we substitute for V from (5) the term involving $(C - A)$ is of the order ε, and in it we may put $r = c$. In the term $\frac{M}{r^2}$, we have $r = c\,(1 + \varepsilon \sin^2\theta)$.

Making these substitutions we get

$$g = \frac{M}{c^2}(1 - 2\varepsilon \sin^2\theta) + \frac{3\,(C - A)}{2c^4}(3\sin^2\theta - 2) - \omega^2 c \sin^2\theta.$$

Substituting in this equation the value obtained in (6) for $\frac{C - A}{Mc^2}$, we have

$$g = \frac{M}{c^2}\left\{1 + q - 2\varepsilon + \left(\varepsilon - \frac{5}{2}q\right)\sin^2\theta\right\}. \qquad (7)$$

If g_p and g_e denote polar and equatorial gravity, respectively, we get

$$g_p - g_e = \left(\frac{5}{2}q - \varepsilon\right)\frac{M}{c^2},$$

and in terms of the order ε we may put $g_p = g_e = \frac{M}{c^2}$; hence we obtain

$$\gamma = \frac{g_p - g_e}{g_e} = \frac{5}{2}q - \varepsilon,$$

which is the same as (4).

From (7) we have $g - g_e = \left(\frac{5}{2}q - \varepsilon\right)\frac{M}{c^2}\cos^2\theta$; whence, if we put $\gamma_\theta = \frac{g - g_e}{g_e}$, we get

$$\gamma_\theta \sec^2\theta + \varepsilon = \frac{5}{2}q. \qquad (8)$$

Examples.

1. If two distributions of mass have the same potential throughout space external to both, their centres of inertia and principal axes are coincident, and their principal moments of inertia are equal, respectively.

If each potential be expanded as in Art. 78, the two series must be equal for all values of r greater than a certain limit, and for all directions. Equating then the coefficients of $\frac{1}{r^2}$ in the two series and also those of $\frac{1}{r^3}$, we obtain the results required.

2. If a body be centrobaric, its baric centre coincides with its centre of inertia, and its moments of inertia round all axes through this point are equal.

3. Find the components of the Earth's attraction, at a point on its surface, parallel and perpendicular to the line joining this point to the centre.

If the components required be denoted by R and P, we have $R = -\frac{dV}{dr}$; whence, by Art. 81, we get,

$$R = \frac{M}{c^2}(1 - 2\epsilon \sin^2\theta) + \frac{3(C - A)}{2c^4}(3\sin^2\theta - 2)$$

$$= \frac{M}{c^2}\left\{1 + q - 2\epsilon + \left(\epsilon - \frac{3}{2}q\right)\sin^2\theta\right\}.$$

Again, $$P = \frac{dV}{rd\theta}.$$

Hence, neglecting small quantities of the second order, by (5) we have

$$P = \frac{3(C - A)}{r^4}\sin\theta\cos\theta = \frac{3(C - A)}{c^4}\sin\theta\cos\theta = \frac{M}{c^2}\frac{3(C - A)}{Mc^2}\sin\theta\cos\theta$$

$$= 2\frac{M}{c^2}\left(\epsilon - \frac{q}{2}\right)\sin\theta\cos\theta = \frac{M}{c^2}\left(\epsilon - \frac{q}{2}\right)\sin 2\lambda,$$

where λ is the latitude of the place.

The direction of P is towards the equator.

4. A rigid body having its centre of inertia fixed is attracted by a distant homogeneous sphere; find the equation of the potential ellipsoid which determines the small oscillations of the body about its position of stable equilibrium. (*See* "Dynamics," Art. 320, Ex. 6.)

Let m be the mass of the sphere; r the distance, and x, y, z, the coordinates of its centre P, referred to the centre of inertia and principal axes of the body; then N_1, N_2, N_3 are given by equations (3), m being put for L in those equations. If now the body receive small rotations θ, ϕ, ψ round its principal axes, the coordinates of P are altered in the same way as if it received equal and opposite rotations, and the body remained fixed. Hence

$$\delta x = y\psi - z\phi, \quad \delta y = z\theta - x\psi, \quad \delta z = x\phi - y\theta,$$

also, since δr is zero, N_1 becomes $N_1 + L$, where

$$L = \frac{3m}{r^5}(C-B)\{y(x\phi - y\theta) + z(z\theta - x\psi)\}$$

$$= \frac{3m}{r^5}(B-C)\{(y^2-z^2)\theta - xy\phi + xz\psi\},$$

and as the original position is one of equilibrium, $N_1 = 0$. Similar results hold good for N_2 and N_3. Again, if σ be the magnitude of the rotation which brings the body into its actual position, α, β, γ the direction cosines of its axis, and W the work done by the attraction of the sphere in the displacement,

$$W = \int_0^\sigma (L\alpha + M\beta + N\gamma)\,d\sigma.$$

Substituting their values for L, M, N, and remembering that $\theta = \sigma\alpha$, $\phi = \sigma\beta$, $\psi = \sigma\gamma$, we get by integration $2W = L\theta + M\phi + N\psi$. Since N_1, N_2, N_3 are each zero, so also are the products xy, yz, zx; hence the point P lies on one of the principal axes of the body, and

$$2W = \frac{3m}{r^5}\{(B-C)(y^2-z^2)\theta^2 + (C-A)(z^2-x^2)\phi^2 + (A-B)(x^2-y^2)\psi^2\}.$$

In a position of stable equilibrium W must be negative for all values of θ, ϕ, ψ.
Let $A > B > C$; then if $x = 0$ and $y = 0$,

$$2W = -\frac{3m}{r^5}z^2\{(B-C)\theta^2 + (A-C)\phi^2\}.$$

This is always negative, and the equation of the potential ellipsoid is $(B-C)x^2 + (A-C)y^2 =$ constant, which represents a cylinder. For stable equilibrium, the centre of the sphere must be situated on the production of the axis of least inertia of the body.

If the body be a homogeneous ellipsoid of small ellipticity, the preceding investigation applies even though the sphere be not distant.

5. A rigid body K having its centre of inertia fixed is attracted by a distant immoveable homogeneous sphere; the initial position of K being given, determine the impulsive couple which must act on it in order that its axis of rotation should be invariable.

If K be free, find the position and motion which must be given to it initially in order that it should continue to revolve round the same axis.

If $\omega_1, \omega_2, \omega_3$ be the angular velocities of the body round its principal axes, its equations of motion ("Dynamics," Art. 267) are

$$A\frac{d\omega_1}{dt} - (B-C)\omega_2\omega_3 = N_1 = \frac{3m}{r^5}(C-B)yz, \text{ \&c.};$$

and $\dfrac{d\omega_1}{dt} = 0$, provided $\omega_2\omega_3 = \dfrac{3m}{r^5}yz$.

Let the direction angles of the initial axis of rotation be α, β, γ; then $\omega_2\omega_3 = \omega^2 \cos\beta \cos\gamma$, so that if

$$\omega^2 = \frac{3m}{r^3},\ \cos\beta = \frac{y}{r},\ \cos\gamma = \frac{z}{r},\ \cos\alpha = \frac{x}{r},\quad \text{we have } \frac{d\omega_1}{dt},\ \frac{d\omega_2}{dt},\ \frac{d\omega_3}{dt}$$

initially zero. Also $\dot{x}$, $\dot{y}$, and $\dot{z}$ are initially zero. Hence, the successive differential coefficients of ω_1, ω_2, ω_3, with respect to the time, are all zero initially, and therefore ω_1, ω_2, ω_3 are constant.

If H be the moment of the impulsive couple required, its components round the axes are, therefore,

$$\sqrt{\frac{3m}{r^5}}\,.\,Ax,\qquad \sqrt{\frac{3m}{r^5}}\,.\,By,\qquad \sqrt{\frac{3m}{r^5}}\,.\,Cz,$$

and the invariable axis of rotation is the line joining the centre of inertia of K to the centre of the sphere.

If K be free, let G denote its centre of inertia, and O the centre of the sphere; then, if GO be made to coincide with a principal axis of K, and if K be projected with a velocity $\sqrt{\frac{m}{r}}$ along another principal axis, and be given an angular velocity $\sqrt{\frac{m}{r^3}}$ round the third, one principal axis of K will always be directed towards O, and K will continue to rotate uniformly round an axis perpendicular to its plane of motion, whilst G describes in this plane a circle round O as centre with a constant velocity.

6. Find the potential of the mass distributed over a plane area at a distant point P in its plane, the force due to an element of mass varying inversely as the distance.

Let C be the moment of inertia of M round the principal axis perpendicular to its plane at G its centre of inertia, r the distance of P from G, and I the moment of inertia of M round GP; then V, the potential at P, is given by the equation

$$V = M \log\frac{1}{r} + \frac{C - 2I}{2r^2}\,.$$

7. If the potential energy due to the mutual action of two invariable mass systems, A and B, be zero for all positions of B outside A, and if the total mass of B be not zero, show that at all points outside the system A its potential is zero.

If V denote the potential of A at a point P in B whose coordinates are x, y, z, at any other point Q, whose coordinates relative to P are ξ, η, ζ, the potential of A is

$$V + \xi\frac{dV}{dx} + \eta\frac{dV}{dy} + \zeta\frac{dV}{dz} + \&c.,$$

and if ρ denote the density of B at Q, and W the energy due to the mutual action of the systems A and B, we have

$$W = \int\left\{V + \xi\frac{dV}{dx} + \eta\frac{dV}{dy} + \zeta\frac{dV}{dz} + \&c.,\right\}\rho\,d\xi\,d\eta\,d\zeta.$$

Hence

$$W = mV + a\frac{dV}{dx} + b\,\frac{dV}{dy} + c\,\frac{dV}{dz} + e\,\frac{d^2V}{dx^2} + \&c.,$$

where m denotes the mass of B, and a, b, &c., are quantities depending on the position of P in the system B, but independent of its position in space. For all positions of P more remote from the origin than the most distant point of A the potential V can, by Art. 78, be expanded in a series of descending powers of r, so that

$$V = \frac{M}{r} + \Sigma\,\frac{Y_n}{r^{n+1}},$$

where M denotes the mass of A, and Y_n is a function of the angular coordinates of the point P.

If this expression for V be substituted in W, it becomes a series in descending powers of r; and since W is zero for all positions of B outside A, the coefficients, in this series, of the different powers of r must be each zero. The process of operating with $\frac{d}{dx}$, $\frac{d}{dy}$, or $\frac{d}{dz}$ on V, or any of its differential coefficients, diminishes by unity the exponent of each power of r in the expansion of the function. Hence, $\frac{mM}{r}$ is the term in W containing the lowest power of r^{-1}, and therefore $M = 0$. The first term in V now becomes $\frac{Y_1}{r^2}$ and, consequently, $\frac{mY_1}{r^2}$ is the term in W containing the lowest power of r^{-1}; whence $Y_1 = 0$. Proceeding in a similar manner with respect to Y_2, &c., we find that each term of V is zero.

The theorem above is due to Mr. F. Purser.

8. Prove Thomson's Theorem, Ex. 20, Art. 75, by the method of the last Example.

If we take as origin the point in A through which the resultant force passes, if V denote the potential of A at any point x, y, z, and m the mass of B, and if we put

$$x\,\frac{dV}{dy} - y\,\frac{dV}{dx} = U;$$

we have $\int U dm = 0$ for all positions of B. Then, as U can be expanded in descending powers of r, by a process similar to that employed in the last Example we find that $U = 0$, and as the axis of z may have any direction, we conclude that A is centrobaric.

This proof of Thomson's Theorem is due to Mr. F. Purser.

CHAPTER V.

SURFACES AND CURVES OF THE SECOND DEGREE.

82. **Introductory.**—The attraction of a homogeneous ellipsoid at a point on its surface was investigated in Arts. 21, 22; and in Art. 75, Ex. 5, it was shown that on the result of this investigation could be based a method of finding the attraction of an ellipsoid in external space.

This problem is one of great celebrity in the history of Mathematics, and has been solved by various methods, of which the most celebrated are those of Mac Claurin, Chasles, Ivory, and Thomson.

A number of expressions for the potential of an ellipsoid have been given by mathematicians of eminence; and in consequence of its connexion with the theory of columnar vortices in a perfect liquid, the determination of the uniplanar potential of a homogeneous elliptic plate is a question of much interest.

The distribution of electricity on conductors whose surfaces are hyperboloids or paraboloids has been treated by Maxwell, following Lamé, by means of elliptic coordinates.

It is proposed in the present chapter to give some account of the results enumerated above.

83. **Surface Distribution Equivalent to Solid Ellipsoid.**—If we assume a distribution of mass such that the potential V is zero at every point outside the surface of the ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and that

$$V = K\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)$$

at every point inside this surface, where K is a constant; then

V is continuous and satisfies the differential equation $\nabla^2 V = 0$ in external space, and the equation

$$\nabla^2 V + 2K\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = 0$$

throughout the interior of the ellipsoid. Also at the surface $\frac{dV}{dx}$, $\frac{dV}{dy}$, and $\frac{dV}{dz}$ change discontinuously. Hence the potential V is due to a homogeneous volume distribution throughout the ellipsoid conjoined with a surface distribution whose effect in external space is equal and opposite to that of the former.

To determine the density σ of the latter distribution, let $-l$, $-m$, $-n$ be the direction cosines of the normal ν drawn inwards at any point of the ellipsoid, and p the central perpendicular on the tangent plane; then, since $\frac{l}{p} = \frac{x}{a^2}$, &c., we have

$$-4\pi\sigma = \frac{dV}{d\nu} = -\left(l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz}\right)$$

$$= 2K\left(l\frac{x}{a^2} + m\frac{y}{b^2} + n\frac{z}{c^2}\right) = \frac{2K}{p}. \qquad (1)$$

Hence, by Ex. 2, Art. 75, the density σ at any point is proportional to the thickness at that point of a shell comprised between the given ellipsoid and an infinitely near confocal ellipsoid. Such a shell is called a *focaloid*.

If the surface distribution whose density is σ be reversed, we get a distribution which is equivalent to the solid ellipsoid in external space. (See Art. 64.)

We have therefore Thomson's Theorem, viz. :—

The potential of a homogeneous ellipsoid in external space is the same as that of a focaloid of equal mass coinciding with its surface.

84. **Thomson's Proof of MacClaurin's Theorem.**— If we suppose the mass of a focaloid to be uniformly distributed throughout the space bounded by its internal surface the potential in external space is thereby unaltered. Hence,

if a homogeneous ellipsoid be diminished in size, but increased in density, by removing a focaloidal stratum of mass from its exterior, and distributing this mass uniformly through the remainder of the space occupied by the ellipsoid, the potential in external space is unchanged. This process may be repeated *ad infinitum*, and therefore we conclude that:

Confocal ellipsoids of equal mass have the same potential in space external to both.

85. **Attraction of Ellipsoid at External Point.**—If $-x$, $-y$, $-z$ be the coordinates of the point P outside the homogeneous ellipsoid whose axes are $2a$, $2b$, $2c$, and whose mass is M, and if $2a'$, &c. be the axes, and M' the mass of the confocal ellipsoid passing through P, the components X', Y', Z' of the attraction of M' at P are given by equations (15) Art. 21.

By Art. 84 the components X, Y, Z of the attraction of M at P are connected with X', Y', Z' by the equations

$$X = \frac{M}{M'} X', \qquad Y = \frac{M}{M'} Y', \qquad Z = \frac{M}{M'} Z'.$$

Hence

$$X = \frac{3Mx}{c'^3} \int_0^1 \frac{u^2 du}{(1 + \lambda'^2_1 u^2)^{\frac{3}{2}} (1 + \lambda'^2_2 u^2)^{\frac{1}{2}}}.$$

If we change the variable under the integral sign by assuming $\frac{v}{c} = \frac{u}{c'}$, we get

$$\lambda'^2_1 u^2 = \lambda_1^2 v^2, \quad \lambda'^2_2 u^2 = \lambda_2^2 v, \quad \frac{u^2 du}{c'^3} = \frac{v^2 dv}{c^3},$$

and the limits of the integral become $\frac{c}{c'}$ and 0; whence

$$X = \frac{3Mx}{c^3} \int_0^{\frac{c}{c'}} \frac{v^2 dv}{(1 + \lambda_1^2 v^2)^{\frac{3}{2}} (1 + \lambda_2^2 v^2)^{\frac{1}{2}}}.$$

Y and Z are obtained in a similar manner.

The components of the attraction of a homogeneous

ellipsoid at an external point, whose coordinates are $-x, -y, -z$, are given therefore by the equations:

$$\left.\begin{aligned} X &= \frac{3\,Mx}{c^3}\int_0^{\frac{c}{c'}} \frac{u^2du}{(1+\lambda_1^{\,2}\,u^2)^{\frac{3}{2}}\,(1+\lambda_2^{\,2}\,u^2)^{\frac{1}{2}}} \\ Y &= \frac{3\,My}{c^3}\int_0^{\frac{c}{c'}} \frac{u^2du}{(1+\lambda_1^{\,2}\,u^2)^{\frac{1}{2}}\,(1+\lambda_2^{\,2}\,u^2)^{\frac{3}{2}}} \\ Z &= \frac{3\,Mz}{c^3}\int_0^{\frac{c}{c'}} \frac{u^2du}{(1+\lambda_1^{\,2}\,u^2)^{\frac{1}{2}}\,(1+\lambda_2^{\,2}\,u^2)^{\frac{1}{2}}} \end{aligned}\right\}. \qquad (2)$$

These expressions differ from those of (15), Art. 21, only in the upper limit of the integral which occurs in each.

Since M, h, and k are the same for two confocal ellipsoids of equal mass, if ψ' be substituted for ψ_1 in equations (27), Art. 24, where $\tan\psi' = \frac{k}{c'}$, we obtain

$$\left.\begin{aligned} X &= \frac{3\,Mx}{h^2k}\,\{F(\psi') - E(\psi')\} \\ Y &= 3\,My\left\{\frac{kE(\psi')}{h^2\,(k^2-h^2)} - \frac{F(\psi')}{h^2k} - \frac{\sin\psi'\,\cos\psi'}{k\,(k^2-h^2)\,\Delta\,(\psi')}\right\} \\ Z &= \frac{3\,Mz}{k\,(k^2-h^2)}\,\{\tan\psi'\,\Delta\,(\psi') - E\,(\psi')\} \end{aligned}\right\}. \qquad (3)$$

It is to be observed that c' is a function of x, y, z, and therefore that the components of the attraction of an ellipsoid at an external point are complicated functions of the coordinates, but at an internal point are linear functions, as we have seen in Art. 21.

86. **Potential of Ellipsoid.**—If V be the potential of a homogeneous ellipsoid at an internal point whose coordinates are x, y, z, by (15), Art. 21, we have

$$\frac{dV}{dx} = -Ax, \qquad \frac{dV}{dy} = -By, \qquad \frac{dV}{dz} = -Cz.$$

Integrating we get

$$V = V_0 - \tfrac{1}{2}(Ax^2 + By^2 + Cz^2), \tag{4}$$

where V_0 is the value of V at the centre of the ellipsoid. To find the value of V_0 we have

$$V_0 = \iiint \frac{\rho r^2 dr d\omega}{r} = \tfrac{1}{2}\rho \int_0^\pi \int_0^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{\rho}{2}\int_0^\pi \int_0^{2\pi} \frac{a^2b^2c^2 \sin\theta \, d\theta \, d\phi}{a^2b^2\cos^2\theta + c^2\sin^2\theta\,(a^2\sin^2\phi + b^2\cos^2\phi)}.$$

Treating the integral in a manner similar to that employed in Art. 21, we get

$$V_0 = \frac{3M}{2c}\int_0^1 \frac{du}{(1+\lambda_1^2u^2)^{\frac{1}{2}}(1+\lambda_2^2u^2)^{\frac{1}{2}}}. \tag{5}$$

If, as in Art. 24, we put $\lambda_1 u = \tan\psi$, we obtain

$$V_0 = \frac{3M}{2k} F(\kappa, \psi_1), \tag{6}$$

where $\kappa = \frac{h}{k}$, and $\tan\psi_1 = \lambda_1$.

Substituting their values for V_0, A, B, C in (4), we find

$$V = \frac{3M}{2c^3}\int_0^1 \left\{c^2 - \frac{x^2u^2}{1+\lambda_1^2u^2} - \frac{y^2u^2}{1+\lambda_2^2u^2} - z^2u^2\right\}\frac{du}{(1+\lambda_1^2u^2)^{\frac{1}{2}}(1+\lambda_2^2u^2)^{\frac{1}{2}}}. \tag{7}$$

If the integrals in (7) be expressed by elliptic functions, we have

$$V = \frac{3M}{2k}F(\psi_1) - \tfrac{1}{2}(Ax^2 + By^2 + Cz^2), \tag{8}$$

where A, B, and C, are the coefficients of x, y, and z in equations (27), Art. 24.

To find the potential of a homogeneous ellipsoid at an external point x, y, z, we suppose a confocal ellipsoid whose

smallest axis is $2c'$, to pass through the point and proceed as in Art. 85. In this manner we obtain

$$V = \frac{3M}{2c^3}\int_0^{\frac{c}{c'}} \left\{c^2 - u^2\left(\frac{x^2}{1+\lambda_1{}^2u^2} + \frac{y^2}{1+\lambda_2{}^2u^2} + z^2\right)\right\} \frac{du}{(1+\lambda_1{}^2u^2)^{\frac{1}{2}}(1+\lambda_2{}^2u^2)^{\frac{1}{2}}}. \quad (9)$$

If we desire to express the potential at an external point by means of elliptic functions, we may use equation (8), substituting in that equation ψ' for ψ_1, where $c' \tan\psi' = k$.

87. **Symmetrical Expressions for Potential and Components of Attraction.**—By the transformations given in Art. 22, we find that at an internal point x, y, z, the potential V of a homogeneous ellipsoid is given by the equation

$$V = \frac{3M}{4}\int_0^\infty \left(1 - \frac{x^2}{a^2+v} - \frac{y^2}{b^2+v} - \frac{z^2}{c^2+v}\right)\frac{dv}{\sqrt{(a^2+v)(b^2+v)(c^2+v)}}. \quad (10)$$

At an external point if $2a'$, $2b'$, $2c'$ be the axes of the confocal ellipsoid passing through it, we have

$$V = \frac{3M}{4}\int_0^\infty \left(1 - \frac{x^2}{a'^2+v} - \frac{y^2}{b'^2+v} - \frac{z^2}{c'^2+v}\right)\frac{dv}{\sqrt{(a'^2+v)(b'^2+v)(c'^2+v)}}.$$

If we put $a'^2 + v = a^2 + u$, we have $b'^2 + v = b^2 + u$, $c'^2 + v = c^2 + u$, and $dv = du$; also $u = \infty$ when $v = \infty$, but $u = a'^2 - a^2$ when $v = 0$. Hence, if q be the greatest root of the equation

$$\frac{x^2}{a^2+q} + \frac{y^2}{b^2+q} + \frac{z^2}{c^2+q} = 1,$$

we obtain

$$V = \frac{3M}{4}\int_q^\infty \left(1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right)\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \quad (11)$$

In like manner, if X, Y, Z denote the components of the attraction of the ellipsoid at the point whose coordinates are $-x, -y, -z$, we have

$$\left.\begin{aligned} X &= 2\pi\rho abcx \int_q^\infty \frac{du}{(a^2+u)^{\frac{3}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}} \\ Y &= 2\pi\rho abcy \int_q^\infty \frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{3}{2}}(c^2+u)^{\frac{1}{2}}} \\ Z &= 2\pi\rho abcz \int_q^\infty \frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{3}{2}}} \end{aligned}\right\}. \qquad (12)$$

Examples.

1. Prove that, at an internal point, x, y, z, the potential V of a homogeneous ellipsoid is given by the equation

$$V = \pi\rho abc \left\{ I + 2\frac{dI}{d(a^2)}x^2 + 2\frac{dI}{d(b^2)}y^2 + 2\frac{dI}{d(c^2)}z^2 \right\},$$

where

$$I = \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

This follows from equation (10).

2. Show that the potential of an ellipsoid at an external point may be put into the form

$$\frac{3M}{4}\left\{ I' + 2\frac{dI'}{d(a'^2)}x^2 + 2\frac{dI'}{d(b'^2)}y^2 + 2\frac{dI'}{d(c'^2)}z^2 \right\},$$

where M is the mass of the ellipsoid, a', b', c' the semi-axes of the confocal ellipsoid passing through the point x, y, z, and

$$I' = \int_0^\infty \frac{du}{\sqrt{(a'^2+u)(b'^2+u)(c'^2+u)}}.$$

3. If I have the same meaning as in Ex. 1, show that

$$2\pi abcI = \iint r^2 d\omega,$$

where r is the radius vector from the centre to any element of the surface of the ellipsoid, and $d\omega$ is the solid angle subtended at the centre by the element.

If V_0 be the potential of the ellipsoid at its centre, we have

$$\pi\rho abcI = V_0 = \iiint \frac{\rho r^2\,dr\,d\omega}{r} = \frac{\rho}{2}\iint r^2\,d\omega.$$

4. Prove that

$$a^2\frac{dI}{d(a^2)} + b^2\frac{dI}{d(b^2)} + c^2\frac{dI}{d(c^2)} = -\tfrac{1}{2}I.$$

From the equation of the ellipsoid we have

$$\frac{1}{r^2} = \frac{\cos^2\theta}{c^2} + \frac{\sin^2\theta\,\cos^2\phi}{a^2} + \frac{\sin^2\theta\,\sin^2\phi}{b^2},$$

and $$\iint r^2\,d\omega = \int_0^{\pi}\int_0^{2\pi} r^2 \sin\theta\,d\theta\,d\phi.$$

Hence $$\frac{1}{abc}\iint r^2\,d\omega$$

is a homogeneous function of a^2, b^2, and c^2 of the degree $-\frac{1}{2}$.

5. Prove that

$$2(a^2-b^2)\frac{d^2I}{d(a^2)\,d(b^2)} = \frac{dI}{d(a^2)} - \frac{dI}{d(b^2)}.$$

6. If the components of the attraction of a homogeneous ellipsoid at an internal point x, y, z, be denoted by $-Ax$, $-By$, $-Cz$, show that

$$A\frac{da}{a} + B\frac{db}{b} + C\frac{dc}{c}$$

is a perfect differential.

Here $$A = -4\pi\rho abc\,\frac{dI}{d(a^2)},\ \text{\&c.}$$

Hence, if we put ξ, η, ζ for a^2, b^2, c^2, we have to show that

$$\sqrt{\left(\frac{\eta\zeta}{\xi}\right)}\frac{dI}{d\xi}d\xi + \sqrt{\left(\frac{\zeta\xi}{\eta}\right)}\frac{dI}{d\eta} + \sqrt{\left(\frac{\xi\eta}{\zeta}\right)}\frac{dI}{d\zeta}$$

is a perfect differential; but this follows immediately from Ex. 5, by which

$$2(\xi-\eta)\frac{d^2I}{d\xi\,d\eta} = \frac{dI}{d\xi} - \frac{dI}{d\eta}.$$

7. Find the potential of a homogeneous focaloid at an internal point.

If V be the potential of a homogeneous ellipsoid, and U that of a focaloid of equal mass having the surface of the ellipsoid as its boundary, by Art. 83, at an internal point x, y, z, we have

$$V - U = K\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right),$$

where

$$K = \frac{2\pi\rho a^2b^2c^2}{a^2b^2+b^2c^2+c^2a^2} = \frac{3}{2}M\frac{abc}{a^2b^2+b^2c^2+c^2a^2},$$

the mass of the focaloid being M.

Hence $$U = V_0 - K + \left(\frac{K}{a^2} - \frac{A}{2}\right)x^2 + \left(\frac{K}{b^2} - \frac{B}{2}\right)y^2 + \left(\frac{K}{c^2} - \frac{C}{2}\right)z^2,$$

where the values of V_0, A, B, and C are given by (5), Art. 86, and (15), Art. 21.

8. Find the components of the force produced at any point inside an ellipsoid by a distribution of mass on its surface whose density σ at any point x, y, z is given by the equation $\sigma = p\,(Lx + My + Nz)$, where p is the central perpendicular on the tangent plane at the point, and L, M, N are constants.

Imagine a homogeneous solid ellipsoid E bounded by the given surface to receive a small translational displacement whose components parallel to the axes of its initial position are $\epsilon_1, \epsilon_2, \epsilon_3$. The perpendicular distance of the origin, or initial position of the centre, from the tangent plane at the point whose initial coordinates are x, y, z, becomes then

$$p + \epsilon_1\frac{px}{a^2} + \epsilon_2\frac{py}{b^2} + \epsilon_3\frac{pz}{c^2},$$

where p is the value of this perpendicular before the displacement. Hence the normal thickness δp of the shell comprised between the two positions of the surface of the ellipsoid is given by the equation

$$\delta p = p\left\{\frac{\epsilon_1}{a^2}x + \frac{\epsilon_2}{b^2}y + \frac{\epsilon_3}{c^2}z\right\}.$$

The density σ of a surface distribution equivalent to the shell is $\rho\delta p$; and by making ρ, the volume density of E, sufficiently great and assigning proper values to $\epsilon_1, \epsilon_2, \epsilon_3$, we can satisfy the equations

$$\frac{\rho\epsilon_1}{a^2} = L, \qquad \frac{\rho\epsilon_2}{b^2} = M, \qquad \frac{\rho\epsilon_3}{c^2} = N.$$

The components X, Y, Z, of the force exercised at the point P by the surface distribution of attractive mass of density σ are now seen to be the changes in the components of the attraction of E at P, when P receives displacements $-\epsilon_1, -\epsilon_2$, and $-\epsilon_3$. Hence $X = A\epsilon_1$, $Y = B\epsilon_2$, $Z = C\epsilon_3$, where A, B, C are given by (15) Art. 21.

9. The density σ of a distribution of attractive mass is given at any point x, y, z of the surface of an ellipsoid by the equation $\sigma = pf(xyz)$, where f denotes a homogeneous quadratic function, and p the perpendicular from the centre on the tangent plane at the point x, y, z: find the components of the attraction of this mass at any point inside the ellipsoid.

Suppose the solid ellipsoid E bounded by the given surface to receive small angular displacements $\delta\theta, \delta\phi, \delta\psi$, round its axes, and let E' be its new position.

If α, β, γ denote the direction cosines of the perpendicular on the tangent plane to E at the point x, y, z, we have $p^2 = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2$. If p' denote the perpendicular on the parallel tangent plane to E', the direction cosines of p' referred to the axes of E' are $\alpha + \delta\alpha$, &c., where $\delta\alpha = \beta\delta\psi - \gamma\delta\phi$, &c. ('Dynamics' (7), Art. 255), then $p' = p + \delta p$, where

$$p\delta p = a^2\alpha\delta\alpha + b^2\beta\delta\beta + c^2\gamma\delta\gamma.$$

The thickness of the shell comprised between

E and E' is δp;

also, we have $\alpha = \frac{px}{a^2}$, &c., and therefore

$$\frac{\delta p}{p} = x\left(\frac{y}{b^2}\delta\psi - \frac{z}{c^2}\delta\phi\right) + y\left(\frac{z}{c^2}\delta\theta - \frac{x}{a^2}\delta\psi\right) + z\left(\frac{x}{a^2}\delta\phi - \frac{y}{b^2}\delta\theta\right)$$

$$= \frac{a^2 - b^2}{a^2b^2}xy\delta\psi + \frac{b^2 - c^2}{b^2c^2}yz\delta\theta + \frac{c^2 - a^2}{c^2a^2}zx\delta\phi.$$

Again, suppose each semi-axis of the ellipsoid E to receive a small increment, then the thickness δp of the shell comprised between the ellipsoid thus generated and E is given by the equation

$$p\delta p = \alpha^2 a\delta a + \beta^2 b\delta b + \gamma^2 c\delta c;$$

whence

$$\frac{\delta p}{p} = \frac{x^2}{a^3}\,\delta a + \frac{y^2}{b^3}\,\delta b + \frac{z^2}{c^3}\,\delta c.$$

If we now superpose the two shells, for the total thickness δp we get

$$\delta p = p\left\{\frac{\delta a}{a^3}\,x^2 + \frac{(a^2 - b^2)\,\delta\psi}{a^2 b^2}\,xy + \&\text{c.}\right\};$$

hence $\rho\delta p$ can be identified with the given form for σ.

The components of the force due to E at an internal point x, y, z are $-Ax, -By, -Cz$, and those due to E' are $-A(x + \delta x) + By\delta\psi - Cz\delta\phi$, &c., where $\delta x = y\delta\psi - z\delta\phi$. Hence the component X, parallel to the primary axis of E, due to the attraction of the superposed shells, is given by the equation

$$X = -(A - B)\,y\delta\psi + (A - C)\,z\delta\phi - \left(\frac{dA}{da}\,\delta a + \frac{dA}{db}\,\delta b + \frac{dA}{dc}\,\delta c\right)\,x,$$

and similar equations hold good for Y and Z. The quantities $\rho\delta a$, $\rho\delta\theta$, &c. being already known in terms of the coefficients of $f(x, y, z)$, the forces X, Y, Z are determined as linear functions of the coordinates.

10. If a concentric ellipsoidal cavity be cut out of a homogeneous sphere, find the equipotential surfaces in the interior of the cavity.

The force at any point inside the cavity is the resultant of that due to an attracting sphere and that due to a repelling ellipsoid of equal density. Hence if X, Y, Z be the components of this force,

$$X = -\tfrac{4}{3}\pi\rho x + Ax$$

and therefore by (22), Art. 24,

$$X = \{A - \tfrac{1}{3}(A + B + C)\}\,x = (2A - B - C)\,\frac{x}{3},$$

$$Y = (2B - C - A)\,\frac{y}{3}, \quad Z = (2C - A - B)\,\frac{z}{3};$$

and the equipotential surfaces are given by the equation

$$(2A - B - C)\,x^2 + \&\text{c.} = \text{constant.}$$

11. If a homogeneous ellipsoid E be divided into two parts by any plane P, show that the mutual action between the parts is reducible to a single force, and find its amount.

If E_1 and E_2 denote the portions into which E is divided by P, the force and couple produced by the attraction of E_2 on E_1, are the same as those produced by the attraction of E on E_1, since the resultant force and couple due to the attraction of E_1 on itself are each zero. Let X, Y, Z be the components of the resultant force, and L, M, N those of the resultant couple, then

$$X = -A\int x\,dm, \qquad Y = -B\int y\,dm, \qquad Z = -C\int z\,dm,$$

$$L = (B - C)\int yz\,dm, \qquad M = \int (C - A)\int zx\,dm, \qquad N = (A - B)\int xy\,dm,$$

where the integrals are taken throughout the entire volume of E_1. Hence putting

$$J = -(LX + MY + NZ),$$

we have

$$\frac{J}{\rho^2} = A(B-C)\int x d\mathfrak{S}\int yz d\mathfrak{S} + B(C-A)\int y d\mathfrak{S}\int zx d\mathfrak{S} + C(A-B)\int z d\mathfrak{S}\int xy d\mathfrak{S},$$

where $d\mathfrak{S}$ denotes an element of the volume of E_1. Assume

$$\frac{x}{a} = \frac{\xi}{R}, \qquad \frac{y}{b} = \frac{\eta}{R}, \qquad \frac{z}{c} = \frac{\zeta}{R};$$

then, when the point x, y, z is on the ellipsoid E, the point ξ, η, ζ is on the sphere S whose radius is R; and when the point x, y, z is on the plane P whose equation is $Fx + Gy + Hz + K = 0$, the point ξ, η, ζ is on the plane Q whose equation is

$$\frac{Fa}{R}\xi + \frac{Gb}{R}\eta + \frac{Hc}{R}\zeta + K = 0$$

and we have

$$\frac{J}{\rho^2}\frac{R^9}{a^3b^3c^3} = A(B-C)\int \xi d\Omega \int \eta\zeta d\Omega + \&c.,$$

where $d\Omega$ is an element of the volume of the portion of S cut off by Q. Transform the axes of ξ, η, ζ to a perpendicular ζ' to the plane Q and two other perpendicular axes parallel to Q, then

$$\xi = \alpha_1\xi' + \alpha_2\eta' + \alpha_3\zeta', \quad \eta = \beta_1\xi' + \beta_2\eta' + \beta_3\zeta', \quad \zeta = \gamma_1\xi' + \gamma_2\eta' + \gamma_3\zeta';$$

and as $\quad \int \xi'\delta\Omega = \int \eta' d\Omega = 0, \qquad$ and also $\quad \int \xi'\eta' d\Omega = \int \eta'\zeta' d\Omega = \int \zeta'\xi' d\Omega = 0,$
we have

$$\begin{aligned}\frac{J}{\rho^2}\frac{R^9}{a^3b^3c^3} = \left(\int \zeta' d\Omega\right)\{&A(B-C)\alpha_3\int(\beta_1\gamma_1\xi'^2 + \beta_2\gamma_2\eta'^2 + \beta_3\gamma_3\zeta'^2)\,d\Omega \\ &+ B(C-A)\beta_3\int(\gamma_1\alpha_1\xi'^2 + \gamma_2\alpha_2\eta'^2 + \gamma_3\alpha_3\zeta'^2)\,d\Omega \\ &+ C(A-B)\gamma_3\int(\alpha_1\beta_1\xi'^2 + \alpha_2\beta_2\eta'^2 + \alpha_3\beta_3\zeta'^2)\,d\Omega\}.\end{aligned}$$

The coefficient of $\int \zeta'^2 d\Omega$ in the expression inside the bracket is obviously zero, and we find for the remaining terms

$$AB\,\{\gamma_1(\alpha_3\beta_1 - \beta_3\alpha_1)\int \xi'^2 d\Omega + \gamma_2(\alpha_3\beta_2 - \beta_3\alpha_2)\int \eta'^2 d\Omega\} + \&c.,$$

but $\quad \alpha_3\beta_1 - \beta_3\alpha_1 = \gamma_2, \quad$ and $\quad \alpha_3\beta_2 - \beta_3\alpha_2 = -\gamma_1, \quad$ also $\quad \int \xi'^2 d\Omega = \int \eta'^2 d\Omega;$

hence the coefficient of AB is zero. In like manner the coefficients of BC and CA are each zero; hence $J = 0$, which is the well known condition that the force and couple should be reducible to a single force. The components of this force are $-mAx_1$, $-mBy_1$, and $-mCz_1$, where x_1, y_1, z_1 are the coordinates of the centre of inertia of E_1, and m is its mass.

12. A homogeneous ellipsoid is divided into two parts by a plane perpendicular to an axis: find the mutual attraction between the parts.

Let the plane be perpendicular to the axis of z at a distance q from the centre, then if Z be the required force, we have

$$Z = \rho C\int z\,dxdydz = \frac{\rho C}{2}\left\{\int z^2\,dxdy - q^2\,\Sigma\right\},$$

where Σ is the area of the ellipse in which the plane meets the ellipsoid. We may assume

$$\frac{x}{a} = \sin\theta\cos\phi, \qquad \frac{y}{b} = \sin\theta\sin\phi, \qquad \frac{z}{c} = \cos\theta,$$

then

$$Z = \frac{\rho C}{2}\left\{abc^2\int_0^{\theta_1}\int_0^{2\pi}\cos^3\theta\sin\theta\,d\theta\,d\phi - \pi q^2 a_1 b_1\right\},$$

where $$\cos\theta_1 = \frac{q}{c}, \quad \text{and} \quad a_1 b_1 = \frac{ab(c^2 - q^2)}{c^2}$$

whence

$$Z = \frac{\pi\rho}{4}\,abC\,\frac{(c^2 - q^2)^2}{c^2} = \frac{\rho Cc^3}{4\pi abc}\Sigma^2$$

13. Prove that the attraction of a homogeneous ellipsoid E on a cubical or spherical portion E_1 of its own mass is the same as if the mass of E_1 were concentrated at its own centre of inertia.

If ξ, η, ζ denote the coordinates of any point relative to the centre of inertia P_1 of E_1, and x_1, y_1, z_1 those of P_1 relative to the centre of the ellipsoid, and if L, M, N denote the moments of the attractive forces round the axes meeting at P_1, we have

$$L = B\int\zeta y\,dm - C\int\eta z\,dm = B\int\zeta(y_1+\eta)\,dm - C\int\eta(z_1+\zeta)\,dm = (B-C)\int\eta\zeta\,dm = 0.$$

In like manner $M = 0$, $N = 0$; hence the proposition is obvious.

88. **Uniplanar Potential of Ellipse.**—By a method similar to that employed in Art. 83, a distribution of uniplanar mass can be determined whose potential is zero at all points in the plane of distribution which are external to an ellipse, and equal to $k\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$ at all internal points, the equation of the ellipse being $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Since the potential is zero at infinity, the total mass is zero (Art. 44), and, as in Art. 83, we find that:—

The uniplanar potential of a homogeneous ellipse at any external point is equal to that of a focaloidal band of equal uniplanar mass whose boundary coincides with that of the ellipse.

It is then easy to prove, as in Art. 84, that—

Confocal ellipses of equal uniplanar mass have the same potential at all points in their plane which are external to both.

From this it follows that the attractions of confocal ellipses of equal mass are equal at all external points.

If X and Y be the components at the point P, whose coordinates are x and y, of the force due to the ellipse E, whose uniplanar mass is M, and whose semi-axes are a and b, by Art. 19, we have, therefore,

$$X = \frac{2M}{a' + b'}\frac{x}{a'}, \qquad Y = \frac{2M}{a' + b'}\frac{y}{b'}, \qquad (13)$$

where a' and b' are the semi-axes of the confocal ellipse passing through P.

To find the potential V at an external point, we have $V = C - \int (Xdx + Ydy)$. Since $a'^2 - b'^2 = a^2 - b^2 = c^2$ we may assume $a' = c \cosh \eta$, $b' = c \sinh \eta$, and since $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, we may assume $x = a' \cos \xi$, $y = b' \sin \xi$; hence we have

$$x = c \cosh \eta \cos \xi, \quad y = c \sinh \eta \sin \xi;$$

and differentiating we obtain

$$dx = - c \cosh \eta \sin \xi \, d\xi + c \sinh \eta \cos \xi \, d\eta,$$

$$dy = c \sinh \eta \cos \xi \, d\xi + c \cosh \eta \sin \xi \, d\eta,$$

also, we have

$$a' + b' = ce^{\eta}, \qquad \frac{x}{a'} = \cos \xi, \qquad \frac{y}{b'} = \sin \xi;$$

whence, by substitution, we get

$$Xdx + Ydy = M \{ - e^{-2\eta} (\sin 2\xi \, d\xi + \cos 2\xi \, d\eta) + d\eta \}$$

$$= M \{ \tfrac{1}{2} d \, (e^{-2\eta} \cos 2\xi) + d\eta \},$$

and therefore

$$V = C - M \{ \eta + \tfrac{1}{2} e^{-2\eta} \cos 2\xi \}.$$

At a point at an infinite distance from the centre of the confocal system η is infinite and $e^{-2\eta} = 0$, also $a' = \frac{c}{2} e^{\eta}$, and

$$V = M \log \frac{1}{a'} = M (\log 2 - \log c - \eta);$$

hence we get $\qquad C = M (\log 2 - \log c).$

The potential V at an external point is given therefore by the equation

$$V = M\{\log 2 - \log c - \eta - \tfrac{1}{2} e^{-2\eta} \cos 2\xi\}. \qquad (14)$$

At an internal point the potential V is given by the equation

$$V = V_0 - \frac{M}{a+b}\left(\frac{x^2}{a} + \frac{y^2}{b}\right).$$

Hence at the boundary

$$V = V_0 - \frac{M}{2}\{1 + e^{-2\beta} \cos 2\xi\},$$

where $c \cosh \beta = a$. Comparing this with the value of V at the boundary given by (14), we get

$$V_0 = M(\tfrac{1}{2} + \log 2 - \log c - \beta), \qquad (15)$$

and for the potential V at an internal point we obtain the equation

$$V = M\left\{\tfrac{1}{2} + \log 2 - \log c - \cosh^{-1}\frac{a}{c} - \frac{x^2}{a(a+b)} - \frac{y^2}{b(a+b)}\right\}. \qquad (16)$$

EXAMPLES.

Find the uniplanar potential of a homogeneous focaloidal band at an internal point.

If U denote the required potential, M the uniplanar mass of the band, and $2a$, $2b$, and $2c$, its axes, and focal interval, U is given by the equation

$$U = M\left\{\tfrac{1}{2} + \log 2 - \log c - \cosh^{-1}\frac{a}{c} - \frac{ab}{a^2+b^2} + \frac{(a-b)(y^2-x^2)}{(a+b)(a^2+b^2)}\right\}.$$

89. **Confocal Homœoids**—The whole theory of the attraction of ellipsoids has been derived by Chasles from the properties of confocal homœoids, which depend chiefly on the relations between corresponding points.

If there be two coaxal quadrics whose semi-axes are a, b, c, and a', b', c', two points are said to correspond when their

coordinates x, y, z, and x', y', z', referred to the axes of the quadrics, satisfy the equations

$$\frac{x}{a} = \frac{x'}{a'}, \quad \frac{y}{b} = \frac{y'}{b'}, \quad \frac{z}{c} = \frac{z'}{c'}.$$

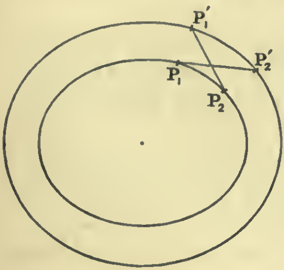
Two coaxal ellipsoidal shells $\mathfrak{S}$ and $\mathfrak{S}'$ are made up of corresponding points if the codirectional axes of the boundaries of $\mathfrak{S}$ are proportional to those of $\mathfrak{S}'$.

To prove this, let a, b, c, be the semi-axes of one boundary of $\mathfrak{S}$, and λa, μb, νc, those of the other, then the semi-axes of the boundaries of $\mathfrak{S}'$ are a', b', c', and $\lambda a'$, $\mu b'$, $\nu c'$, and if we assume

$$\frac{x}{a} = \frac{x'}{a'}, \quad \frac{y}{b} = \frac{y'}{b'}, \quad \frac{z}{c} = \frac{z'}{c'},$$

when the point x, y, z is on a boundary of $\mathfrak{S}$, the point x', y', z' is on the corresponding boundary of $\mathfrak{S}'$.

If any two points, P_1 and P_2, be taken on the ellipsoid whose semi-axes are a, b, c, and the corresponding points P'_1 and P'_2, on the confocal ellipsoid whose semi-axes are a', b', c', the distances $P_1 P'_2$ and $P'_1 P_2$ are equal.



For let x_1, y_1, z_1, be the coordinates of P_1, with a similar notation for the other points, then,

$$P_1P'^2_2 = x_1^2 + y_1^2 + z_1^2 + x'^2_2 + y'^2_2 + z'^2_2 - 2\,(x_1x'_2 + y_1y'_2 + z_1z'_2),$$

but

$$x_1^2 + y_1^2 + z_1^2 = \frac{a^2}{a'^2}\,x_1'^2 + \frac{b^2}{b'^2}\,y'^2_1 + \frac{c^2}{c'^2}\,z'^2_1$$

$$= \frac{a^2 - a'^2 + a'^2}{a'^2}\,x'^2_1 + \frac{b^2 - b'^2 + b'^2}{b'^2}\,y'^2_1 + \frac{c^2 - c'^2 + c'^2}{c'^2}\,z'^2_1$$

$$= a^2 - a'^2 + x'^2_1 + y'^2_1 + z'^2_1.$$

and in like manner

$$x'^2_2 + y'^2_2 + z'^2_2 = a'^2 - a^2 + x_2^2 + y_2^2 + z_2^2,$$

also

$$x_1x'_2 = \frac{a}{a'}x'_1\frac{a'}{a}x_2 = x'_1x_2,$$

and similarly

$$y_1y'_2 = y'_1y_2,\ z_1z'_2 = z'_1z_2;$$

hence by addition we see that $P_1P'_2 = P'_1P_2$.

It is easy to see that, in the theorem above, we may for ellipsoids substitute hyperboloids of the same family.

If there be two thick or thin homœoids, H and H'; such that each boundary of H is confocal with the corresponding boundary of H', the homœoids may be called *doubly confocal*, and the codirectional axes of the boundaries of H are proportional to those of H'.

For the semi-axes of the boundaries of H are a, b, c; $\lambda a, \lambda b, \lambda c$; and those of the boundaries of H' are a', b', c'; $\mu a', \mu b', \mu c'$; then, as

$$a^2 - b^2 = a'^2 - b'^2, \quad \text{and} \quad \lambda^2(a^2 - b^2) = \mu^2(a'^2 - b'^2),$$

we must have $\lambda = \mu$.

We can now show that if P and P' be corresponding points on the surfaces of doubly confocal thin homœoids, H and H', whose masses are M and M', and if $V_{P'}$ be the potential of H at P', and V'_P that of H' at P, then, $V_{P'} : V'_P :: M : M'$.

For, by the preceding part of this Article, the volumes H and H' of the homœoids are composed of corresponding points, and as dH the element of the volume of H at the point x, y, z, is $dxdydz$, we have $\dfrac{dH}{abc} = \dfrac{dH'}{a'b'c'}$; whence, if ρ and ρ' be the volume densities of the homœoids, $\dfrac{\rho dH}{\rho' dH'} = \text{constant} = \dfrac{M}{M'}$.

Again Q being any point in H, and Q' the corresponding point in H', we have

$$V_{P'} = \int\frac{\rho dH}{P'Q}, \qquad \text{and} \qquad V'_P = \int\frac{\rho' dH'}{PQ'},$$

but Q and Q' are infinitely near corresponding points on the confocal ellipsoids passing through P and P', and therefore PQ' and $P'Q$ can differ only by an infinitely small quantity;

hence $V_{P'} : V'_P :: M : M'$. It is plain that this result holds good whatever be the law of force, provided the force due to an element of mass varies as the product of this mass, and some function of its distance.

If the thickness of one homœoid H' be altered, its potential V' and its mass M' are altered in the same ratio. Hence the theorem proved above is true for *any* two confocal homœoids.

It follows from this theorem that the equi-potential surfaces in external space of a homogeneous homœoid are confocal ellipsoids; and also, that at a point external to both, the potentials of confocal homœoids are as their masses.

In fact, if we suppose H' outside H, since a homœoid has the same potential at all internal points, V'_P is constant, and so therefore is $V_{P'}$, whatever be the position of P' on the surface of H'.

Again, if we have two confocal homœoids H and H', through any point P'' outside both we can suppose another confocal homœoid H'' described; then,

$$\frac{V_{P''}}{M} = \frac{V''_P}{M''} = \frac{V''_{P'}}{M''} = \frac{V'_{P''}}{M'}.$$

It is now easy to prove MacClaurin's Theorem by the method given in Ex. 5, Art 75.

90. **Ivory's Theorem.**—If there be two confocal homogeneous solid ellipsoids, E and E', of the same density, whose semi-axes are a, b, c, and a', b', c'; and if $X_{P'}$ be the component parallel to a of the attraction of E at a point P' on the surface of E', and X'_P the parallel component of the attraction of E' at the point P on E corresponding to P', then,

$$\frac{X_{P'}}{bc} = \frac{X'_P}{b'c'}.$$

This theorem is *independent of the law of attraction*, provided the force due to an element of mass is proportional to this mass multiplied by some function of the distance.

To prove Ivory's Theorem, let x, y, z be the coordinates of any point of E, and r' its distance from P', and let $f'(r')$ denote the force due to a unit of mass at the distance r', then

$$X_{P'} = \iiint \rho f'(r') \frac{dr'}{dx}\, dxdydz = \iint \rho f(r')\, dydz,$$

where r' is now the distance from P' of a point on the *surface* of E. In like manner

$$X'_P = \iint \rho f(r)\, dy'\, dz'.$$

To every point x, y, z on the surface of E there corresponds a point x', y', z' on the surface of E', and therefore

$$\frac{dydz}{bc} = \frac{dy'dz'}{b'c'}, \text{ also } \quad r = r';$$

hence the theorem is proved.

By considering three confocal ellipsoids, we can easily deduce MacClaurin's Theorem from that of Ivory, remembering that the component parallel to an axis of the attraction of an ellipsoid at an internal point is proportional to the parallel coordinate of that point, and therefore, in the case of corresponding points, to the parallel semi-axis of the confocal passing through the point.

Examples.

1. Prove by means of Ivory's Theorem that, if a uniform spherical shell exercise no attraction at any internal point, the law of force must be that of the inverse square.

From the assumed hypothesis with respect to a spherical shell, it follows that the attraction of a homogeneous sphere at an internal point is that of the concentric sphere passing through it. Now suppose a sphere S whose radius is R, and let there be three points P, P', P'', on the same straight line through its centre, P being on the surface of S and the others outside it. Describe concentric spheres through P' and P'' whose radii are R' and R'', and let the attractions of these spheres at P be X'_P and X''_P, the attractions of S at P' and P'' being $X_{P'}$ and $X_{P''}$; then

$$\frac{X_{P'}}{R^2} = \frac{X'_P}{R'^2}, \quad \text{and} \quad \frac{X_{P''}}{R^2} = \frac{X''_P}{R''^2},$$

but, as we saw above,

$$X'_P = X''_P, \quad \text{and therefore } X_{P''} = \frac{X_{P'}R'^2}{R''^2};$$

that is, the attraction of S at a point P'' varies inversely as the square of the distance of P'' from the centre of S. By supposing S infinitely small, we obtain the law of force for an element of mass.

2. A hollow insulated conductor, whose inner and outer surfaces S_1 and S_2 are ellipsoids, is filled with non-conducting material containing a quantity of metallic dust charged with electricity and uniformly distributed throughout the material: find the distribution of electricity on the conductor, and the potential in external space.

If the total charge on the metallic dust be E, and if we suppose its action to be the same as that of a homogeneous ellipsoid, the distribution on S_1 is a focaloid whose mass is $-E$, and that on S_2 a homœoid whose mass is E, and the potential in external space is that of the latter.

3. A solid ellipsoid is composed of homogeneous shells bounded by similar surfaces: find its potential at an external point P.

If a, b, c be the semi-axes of the ellipsoid, and ηa, ηb, ηc those of the interior surface of the homœoid whose density is ρ, the potential of this homœoid at P is

$$4\pi abc\rho\eta^2 d\eta \int_{a'}^{\infty} \frac{da'}{b'c'},$$

where a', b', c' are the semi-axes of an ellipsoid confocal with the homœoid which, at the lower limit of the integral, passes through P (Ex. 3, Art. 75). If we put

$$a'^2 = \eta^2 (a^2 + u), \quad b'^2 = \eta^2 (b^2 + u), \quad c'^2 = \eta^2 (c^2 + u),$$

and denote the coordinates of P by x, y, z, we have

$$\int_{a'}^{\infty} \frac{da'}{b'c'} = \int_{v}^{\infty} \frac{du}{2\eta\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

where v, the lower limit of the integral, is given by the equation

$$\frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v} = \eta^2.$$

Hence if V be the potential of the solid ellipsoid at P, we obtain

$$V = 2\pi abc \int_0^1 \rho\eta d\eta \int_v^{\infty} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \qquad (a)$$

If q be the *greatest root* of the equation

$$\frac{x^2}{a^2+q} + \frac{y^2}{b^2+q} + \frac{z^2}{c^2+q} = 1,$$

when $\eta = 1$ we have $v = q$, and also $v = \infty$ when $\eta = 0$; hence, substituting for η in terms of v in (a), we get

$$V = \pi abc \int_{\infty}^{q} \rho \left\{\frac{x^2}{(a^2+v)^2} + \frac{y^2}{(b^2+v)^2} + \frac{z^2}{(c^2+v)^2}\right\} dv \int_{\infty}^{v} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

4. Find the potential of a homogeneous ellipsoid at an external point.

If the ellipsoid be homogeneous, ρ is constant, and integrating by parts the expression for V given by equation (a) in Ex. 3, we get

$$V = \pi\rho abc \left|_0^1 \left\{\eta^2 \int_v^{\infty} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\right\}\right.$$

$$+ \pi\rho abc \int_0^1 \frac{\eta^2 \dfrac{dv}{d\eta} d\eta}{\sqrt{(a^2+v)(b^2+v)(c^2+v)}}.$$

The first term in this expression for V vanishes when $\eta = 0$, and when $\eta = 1$ it becomes

$$\pi\rho abc \int_q^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

In the second term

$$\eta^2 = \frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v},$$

and as before $v = q$ when $\eta = 1$, and $v = \infty$ when $\eta = 0$. Hence finally we obtain

$$V = \pi\rho abc \int_q^\infty \left(1 - \frac{x^2}{a^2+v} - \frac{y^2}{b^2+v} - \frac{z^2}{c^2+v}\right) \frac{dv}{\sqrt{(a^2+v)(b^2+v)(c^2+v)}}.$$

5. If the density ρ at any point Q of an ellipsoid be given by the equation

$$\rho = K\left(\frac{r}{R}\right)^n$$

where r is the distance of Q from the centre, and R the codirectional semi-diameter: find the potential of the ellipsoid at an external point.

In this case, in Ex. 3, we have $\rho = K\eta^n$, and

$$V = 2\pi Kabc \int_0^1 \eta^{n+1}\, d\eta \int_v^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

Integrating by parts, as in Ex. 4, we obtain

$$V = \frac{2\pi Kabc}{n+2} \int_q^\infty \left\{1 - \left(\frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v}\right)^{\frac{n+2}{2}}\right\} \frac{dv}{\sqrt{(a^2+v)(b^2+v)(c^2+v)}}.$$

6. If Γ and Γ' denote the volumes of cones having the centre as vertex and resting on portions, S and S', of the surfaces of quadrics of the same family whose semi-axes are a, b, c, and a', b', c', prove that, if S and S' be composed of corresponding points,

$$\frac{\Gamma}{\Gamma'} = \frac{abc}{a'b'c'}.$$

When the point x, y, z comes on the straight line joining the centre to x_1, y_1, z_1 the point x', y', z' corresponding to x, y, z comes on the straight line joining the centre to x'_1, y'_1, z'_1, corresponding to x_1, y_1, z_1, and when x, y, z comes on S, the corresponding point x', y', z' comes on S'. Hence $\Gamma = \int dxdydz$, and $\Gamma' = \int dx'dy'dz'$, where x', y', z' is always the point corresponding to x, y, z, and therefore

$$\frac{\Gamma}{\Gamma'} = \frac{abc}{a'b'c'}.$$

7. If p and p' be the central perpendiculars on quadric surface elements dS and dS' composed of corresponding points, prove that

$$\frac{pdS}{p'dS'} = \frac{abc}{a'b'c'},$$

where a, b, c and a', b', c' are the semi-axes of the quadrics.

8. If V and V' be the potentials of two doubly confocal hyperboloidal homœoids of the same family whose semi-axes are a, b, c and a', b', c', and whose volume densities are ρ and ρ', and if P and P' be corresponding points on their boundaries, show that $V_{P'} : V'_{P} :: \rho abc : \rho' a'b'c'$.

This is proved in the same manner as the corresponding theorem for ellipsoidal homœoids, Art. 89.

9. Show that for any confocal homœoids of the same family $V_{P'} : V'_{P} :: \epsilon bc : \epsilon' b'c'$, where ϵ and ϵ' are the surface densities at the extremity of the first principal axis on each homœoid.

In the proportion in the preceding Example, we may substitute δa and $\delta a'$ for a and a', but $\epsilon = \rho\delta a$, and $\epsilon' = \rho'\delta a'$; and if δa be altered, ϵ and the potential of the homœoid are altered in the same ratio,

10. An insulated ellipsoidal conductor is charged with electricity; find the total charge on the portion of the ellipsoid cut off by a plane perpendicular to one of the axes.

Let the equation of the plane be $z = f$, and let Q be the charge required. The surface density σ at any point x, y, z on the ellipsoid is given by the equation $\sigma = \frac{\epsilon p}{a}$ where ϵ is the density at the extremity of the axis major, and p the central perpendicular on the tangent plane at x, y, z; if γ be the angle which this perpendicular makes with the axis of z, we have

$$Q = \int \sigma dS = \frac{\epsilon}{a}\int \frac{p}{\cos\gamma}\, dS \cos\gamma = \frac{\epsilon}{a}\int \frac{c^2}{z}\, dxdy.$$

If we assume, as is allowable,

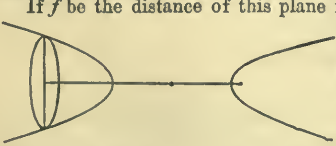
$$x = a\sin\theta\cos\phi, \qquad y = b\sin\theta\sin\phi, \qquad z = c\cos\theta,$$

we get $\quad dxdy = ab\sin\theta\cos\theta\, d\theta d\phi;\quad$ whence we obtain

$$Q = \epsilon bc \iint \sin\theta\, d\theta d\phi = 2\pi\epsilon bc\left(1 - \frac{f}{c}\right).$$

11. Mass is distributed on the surface of a hyperboloid of two sheets so as to form a homœoid; find the total quantity of mass on the portion of one sheet cut off by a plane perpendicular to the first principal axis.

If f be the distance of this plane from the centre, the required quantity Q can be found by a method similar to that in Ex. 10. In the present case we may assume



$$x = a\cosh\theta, \quad y = b\sinh\theta\cos\phi, \quad z = c\sinh\theta\sin\phi,$$

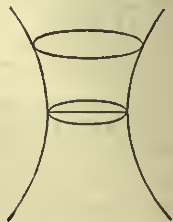
then we obtain $\quad Q = 2\pi\epsilon bc\left(\frac{f}{a} - 1\right).$

Here the equation of the hyperboloid referred to its centre and axes is supposed to be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

12. Find the total mass on the portion of a hyperboloidal homœoid of one sheet intercepted between the plane of xy and a parallel plane.

Here if x, y, z be the coordinates of a point on the surface, we may assume



$$x = a \cosh\theta \cos\phi, \quad y = b \cosh\theta \sin\phi, \quad z = c \sinh\theta,$$

and we find

$$Q = 2\pi\epsilon bc \frac{f}{c} = 2\pi\epsilon bf,$$

where f is the distance of the given plane from the centre, and the equation of the hyperboloid is

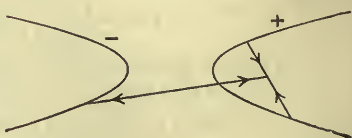
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

It is to be observed that the total mass of the entire homœoid is infinite for the hyperboloid of one sheet, and for each sheet of the hyperboloid of two sheets. In the case of the hyperboloid of two sheets, if one of these sheets be composed of positive mass and the other of negative, the total mass is zero.

13. If one sheet of a hyperboloidal homœoid of two sheets be composed of positive mass and the other of negative, show that the potential of the homœoid at its centre is zero.

14. Show that a hyperboloidal homœoid of two sheets, one positive, the other negative, exercises no attraction at points on the sides of the two sheets remote from the centre.

This is proved by a method similar to that employed in Art. 18.



15. If the equipotential surfaces of a field of force be confocal quadrics of the same family, prove that corresponding points lie on the same line of force.

If a, a', a'' be the primary semiaxes of the three confocals passing through the point x, y, z, by a well-known theorem, Salmon, "Geometry of Three Dimensions," Art. 160,

$$x^2 = \frac{a^2 a'^2 a''^2}{(a^2 - b^2)(a^2 - c^2)},$$

whence if a' and a'' remain unaltered $\frac{x}{a}$ is constant, and similar results hold good for y and z. Hence, corresponding points on ellipsoids confocal with a lie on the intersection of the quadrics a' and a'', which is perpendicular to the ellipsoids, and is therefore a line of force when the ellipsoids are equipotentials.

A similar result is obtained in like manner when the equipotential surfaces are hyperboloids of either family.

16. Prove that an ellipsoidal homœoid exercises equal attractions on corresponding elements of the surfaces of confocal ellipsoids whose surface densities are equal.

These elements are orthogonal sections of the same tube of force, and therefore $\sigma R dS = \sigma R' dS'$, where dS and dS' are the corresponding elements, whose density is σ, and R and R' the resultant forces acting at them.

91. Uniplanar Force varying inversely as the Distance.—The theory of confocal homœoids and corresponding points which has been developed in the preceding Articles for a three dimensional distribution of mass can be established in like manner for a distribution of uniplanar mass acting with a force varying inversely as the distance.

EXAMPLES.

1. In a uniplanar field of force where there is no mass, if the equipotential curves be confocal ellipses, find the potential at any point P.

The lines of force in this case are hyperbolas confocal with the ellipses, and the tubes of force (Ex. 15, Art. 90) intercept corresponding portions on the equipotential curves which cut them orthogonally.

Let ds_1 and ds_2 be the elements of the equipotential ellipses, whose semiaxes are a_1, b_1, and a_2, b_2, intercepted by a tube of force, R_1 and R_2 the resultant forces at ds_1 and ds_2, and p_1 and p_2 the central perpendiculars on the tangents to these elements, then if V be the potential, we have

$$\left(\frac{dV}{dp}\right)_1 = -R_1, \quad \left(\frac{dV}{dp}\right)_2 = -R_2,$$

whence

$$\left(\frac{dV}{dp}\right)_1 ds_1 = \left(\frac{dV}{dp}\right)_2 ds_2;$$

but since ds_1 and ds_2 correspond, as in Ex. 7, Art. 90, we have

$$\frac{p_1\, ds_1}{a_1 b_1} = \frac{p_2\, ds_2}{a_2 b_2},$$

and therefore

$$\left(\frac{dV}{dp}\right)_2 = \frac{a_1 b_1}{a_2 b_2}\frac{p_2}{p_1}\left(\frac{dV}{dp}\right)_1.$$

Taking the intersection of either ellipse with its axis major as one of the corresponding points, we get

$$\left(\frac{dV}{da}\right)_2 = \frac{b_1}{b_2}\left(\frac{dV}{da}\right)_1.$$

If we now suppose a_1 to remain constant and a_2 to vary, we obtain

$$V = b_1 \left(\frac{dV}{da}\right)_1 \int \frac{da_2}{b_2} + \text{constant}.$$

If $2c$ be the focal interval of one of the ellipses of the system, we have $b_2{}^2 = \sqrt{(a_2{}^2 - c^2)}$, and putting $a_2 = c \cosh \eta$, we get $V = i\eta + j$, where i and j are constants, and $c \cosh \eta$ is the semiaxis major of the ellipse of the confocal system passing through P.

2. If the equipotential curves be confocal hyperbolas, find the potential at any point P in the field.

If a' be the primary semiaxis of a hyperbola of the system, proceeding as in the last example, we get

$$V = j - i \int \frac{da'}{\sqrt{(c^2 - a'^2)}},$$

where i and j are constants.

If we put $a' = c \cos \xi$, we have, therefore, $V = i\xi + j$, where $c \cos \xi$ is the primary semiaxis of the hyperbola of the confocal system passing through P.

92. **Poisson's Equation in Elliptic Coordinates.**—Let λ, μ, ν denote the primary semiaxes of the ellipsoid, the hyperboloid of one sheet, and the hyperboloid of two sheets passing through any common point and belonging to a given confocal system.

These surfaces cut at right angles, so that the line of intersection of two surfaces cuts all surfaces of the remaining family perpendicularly. Let s_1, s_2, and s_3 denote the arcs of these curves of intersection which are perpendicular to the ellipsoid, the hyperboloid of one sheet, and the hyperboloid of two sheets respectively, and let α, β, γ be defined by the equations

$$\alpha = \int_k^\lambda \frac{k d\lambda}{\sqrt{(\lambda^2 - k^2)(\lambda^2 - h^2)}}, \quad \beta = \int_h^\mu \frac{k d\mu}{\sqrt{(\mu^2 - h^2)(k^2 - \mu^2)}},$$

$$\gamma = \int_0^\nu \frac{k d\nu}{\sqrt{(h^2 - \nu^2)(k^2 - \nu^2)}}. \qquad (17)$$

where $h^2 = a^2 - b^2$, $k^2 = a^2 - c^2$, the semiaxes of an ellipsoid of the confocal system being denoted by a, b, c.

It was shown, Art. 75, Ex. 2, that, if p be the central perpendicular on a tangent plane to an ellipsoid whose

primary semiaxis is λ, and $p + dp$ that on the parallel tangent plane to the consecutive confocal, $pdp = \lambda d\lambda$, also (Salmon, "Geometry of Three Dimensions," Art. 165)

$$p = \frac{P_1}{D_2 D_3},$$

where P_1 denotes the product of the semiaxes of the ellipsoid, and D_2 and D_3 are the semiaxes of the central section perpendicular to p, whose values are given by the equations

$$D_2^2 = \lambda^2 - \nu^2, \quad D_3^2 = \lambda^2 - \mu^2.$$

If ds_1 be the element of the arc s_1 intercepted between two consecutive confocal ellipsoids,

$$ds_1 = dp = \frac{\lambda D_2 D_3}{P_1} d\lambda = \frac{D_2 D_3}{k} d\alpha;$$

hence
$$\frac{d}{ds_1} = \frac{k}{D_2 D_3} \frac{d}{d\alpha};$$

in like manner,

$$ds_2 = \frac{D_3 D_1}{k} d\beta, \quad \text{and} \quad ds_3 = \frac{D_1 D_2}{k} d\gamma,$$

where $D_1^2 = \mu^2 - \nu^2$; and therefore

$$ds_1 ds_2 ds_3 = \frac{D_1^2 D_2^2 D_3^2}{k^3} d\alpha\, d\beta\, d\gamma.$$

If we now consider the volume comprised between three confocals passing through a point, and three others meeting at another point, as in the case of polar coordinates considered in Art. 48, we have

$$\iiint \nabla^2 V ds_1\, ds_2\, ds_3 = \iint \frac{dV}{ds_1} ds_2 ds_3 + \iint \frac{dV}{ds_2} ds_3\, ds_1$$
$$+ \iint \frac{dV}{ds_3} ds_1 ds_2.$$

Expressing this equation in terms of the variables α, β, γ, we get

$$\iiint \nabla^2 V \frac{D_1^2 D_2^2 D_3^2}{k^3} d\alpha d\beta d\gamma$$

$$= \iint \frac{k}{D_2 D_3} \frac{dV}{d\alpha} \frac{D_1^2 D_2 D_3}{k^2} d\beta d\gamma + \&c.$$

$$= \iiint \left(\frac{D_1^2}{k} \frac{d^2 V}{d\alpha^2} + \frac{D_2^2}{k} \frac{d^2 V}{d\beta^2} + \frac{D_3^2}{k} \frac{d^2 V}{d\gamma^2} \right) d\alpha\, d\beta\, d\gamma,$$

since D_1 is independent of α, D_2 of β, and D_3 of γ. Hence, we have

$$\nabla^2 V = \frac{k^2}{D_1^2 D_2^2 D_3^2} \left\{ D_1^2 \frac{d^2 V}{d\alpha^2} + D_2^2 \frac{d^2 V}{d\beta_2} + D_3^2 \frac{d^2 V}{d\gamma^2} \right\}. \quad (18)$$

Poisson's equation becomes then

$$D_1^2 \frac{d^2 V}{d\alpha^2} + D_2^2 \frac{d^2 V}{d\beta^2} + D_3^2 \frac{d^2 V}{d\gamma^2} + 4\pi\rho \frac{D_1^2 D_2^2 D_3^2}{k^2} = 0; \quad (19)$$

and Laplaces's equation assumes the form

$$D_1^2 \frac{d^2 V}{d\alpha^2} + D_2^2 \frac{d^2 V}{d\beta^2} + D_3^2 \frac{d^2 V}{d\gamma^2} = 0. \quad (20)$$

93. **Determination of the Potential when the Equipotential Surfaces are Confocal Quadrics.**— If a field of force, where there is no mass, be such that the equipotential surfaces are confocal quadrics of the same family, equation (20) enables us to determine the form of the potential throughout the field.

If the equipotentials be ellipsoids, since at all points of the field V is constant when α is constant, V must be independent of β and γ; and therefore by (20), we have

$$\frac{d^2 V}{d\alpha^2} = 0,$$

whence $V = i\alpha + j$, where i and j are constants.

A similar result holds good when the equipotentials are hyperboloids of one sheet or of two sheets.

94. **Confocal Ellipsoids.**—If two confocal ellipsoids, E_1 and E_2, be at constant potentials, A_1 and A_2, the intervening space being unoccupied, and if V be a function of the coordinates which, throughout this space, is equal to $i\alpha + j$, then V satisfies Laplace's equation throughout the field, and if V be equal to A_1 at E_1, and equal to A_2 at E_2, by Art. 70 V must be the potential.

If α_1 and α_2 be the values of α for the surfaces E_1 and E_2, we have

$$i = \frac{A_1 - A_2}{\alpha_1 - \alpha_2} \qquad j = \frac{A_1\alpha_2 - A_2\alpha_1}{\alpha_2 - \alpha_1},$$

whence

$$V = \frac{A_1\alpha_2 - A_2\alpha_1}{\alpha_2 - \alpha_1} - \frac{A_1 - A_2}{\alpha_2 - \alpha_1}\,\alpha. \tag{21}$$

If the ellipsoids are the surfaces of conductors in a state of electric equilibrium, the density σ_1 at any point Q of the inner surface E_1 is, Art. 46, given by the equation

$$4\pi\sigma_1 = -\frac{dV}{ds_1} = -\frac{k}{D_2D_3}\frac{dV}{d\alpha} = \frac{A_1 - A_2}{\alpha_2 - \alpha_1}\frac{kp_1}{a_1b_1c_1}, \tag{22}$$

where $a_1b_1c_1$ are the semi-axes of E_1, and p_1 is the central perpendicular on the tangent plane at Q.

In like manner

$$4\pi\sigma_2 = -\frac{A_1 - A_2}{\alpha_2 - \alpha_1}\frac{kp_2}{a_2b_2c_2}, \tag{23}$$

where σ_2 is the density at any point of the surface E_2.

The distributions on E_1 and E_2 are therefore homœoids whose masses are equal in magnitude but opposite in algebraic sign.

If we suppose E_2 to be at an infinite distance and the potential at its surface to be zero, we have the case of a charged insulated ellipsoidal conductor placed in an infinite field.

It is easy to see that the results which have been obtained for ellipsoidal conductors can without difficulty be arrived at without the use of elliptic coordinates. This will be shown in the Examples.

95. **Hyperboloids of One Sheet.**—When the equipotential surfaces throughout a field in which there is no mass are hyperboloids of one sheet the potential V at any point of the field is given by the equation $V = i\beta + j$.

If the potential have given values B_1 and B_2 at the surfaces H_1 and H_2 at which the values of β are β_1 and β_2, then for the space which lies between H_1 and H_2, we have

$$V = \frac{B_1\beta_2 - B_2\beta_1}{\beta_2 - \beta_1} - \frac{B_1 - B_2}{\beta_2 - \beta_1}\beta. \qquad (24)$$

If corresponding portions of two confocal hyperboloids of one sheet, H_1 and H_2, are the opposite surfaces of conductors in electric equilibrium, and if the remaining boundaries of the field are the portions of a confocal ellipsoid intercepted between the hyperboloids, the resultant force at the ellipsoidal boundary being everywhere tangential to that surface, the potential at any point of the field must be of the form $i\beta + j$, as there can be only one acyclic function of the coordinates which satisfies Laplace's Equation throughout the field as well as the given boundary conditions. If the ellipsoidal boundary of the field be at an infinite distance the opposite boundaries of the conductors are complete hyperboloids of one sheet.

When two such conductors are in electric equilibrium and all the lines of force emanating from one terminate on the other, no lines of force can meet the boundary at infinity except tangentially, and we may conclude, therefore, that the potential at any point between the conductors H_1 and H_2 is of the form $i\beta + j$.

If the equation of the surface H_1 be written in the form

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - \frac{z^2}{c_1^2} = 1,$$

the surface density σ_1 at any point Q of H_1 is given by the the equation

$$4\pi\sigma_1 = \frac{B_1 - B_2}{\beta_2 - \beta_1}\frac{kp_1}{a_1 b_1 c_1}, \qquad (25)$$

where p_1 is the central perpendicular on the tangent plane at Q. For σ_2 the surface density at any point of H_2 we have, in like manner,

$$4\pi\sigma_2 = -\frac{B_1 - B_2}{\beta_2 - \beta_1}\frac{kp_2}{a_2b_2c_2}. \tag{26}$$

Hence the distributions on H_1 and H_2 form hyperboloidal homœoids, and by Ex. 7, Art. 90, the mass on any portion of the surface H_1 is equal in amount, and opposite in algebraical sign, to that on the corresponding portion of H_2.

96. **Hyperboloids of Two Sheets.**—Results similar to those which have been arrived at for hyperboloids of one sheet hold good also for hyperboloids of two sheets.

In this case the surfaces bounding the field may be either the two sheets of the same hyperboloid or sheets of two different hyperboloids.

In the former case, we must suppose $\gamma = \gamma_1$ at one surface, and $\gamma = -\gamma_1$ at the other.

In general, if γ_1 and γ_2 be the values of γ, and C_1 and C_2 those of the potential at the two surfaces bounding the field, the potential V at any intervening point is given by the equation

$$V = \frac{C_1\gamma_2 - C_2\gamma_1}{\gamma_2 - \gamma_1} - \frac{C_1 - C_2}{\gamma_2 - \gamma_1}\gamma. \tag{27}$$

In the case of conductors in electric equilibrium, the surface densities are given by equations which are obtained by putting C and γ instead of B and β in equations (25) and (26), the equations of the surfaces bounding the field being written in the form

$$\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} - \frac{z^2}{c_1^2} = 1, \qquad \frac{x^2}{a_2^2} - \frac{y^2}{b_2^2} - \frac{z^2}{c_2^2} = 1.$$

It will be shown in the Examples that when the equipotential surfaces of a field of force are hyperboloids, the potential can be obtained without the use of elliptic coordinates. If the hyperboloids be of two sheets the potential can be arrived at by means of the properties of homœoids having two sheets such that the surface density is positive at each point of the one, and negative at each point of the other.

A hyperboloidal homœoid having two sheets of which one is composed of positive mass and the other of negative, the volume density irrespective of sign being uniform, may be called *a contrafoliated two-sheeted homœoid.*

97. **Determination of α, β, γ, as Elliptic Functions.** By equations (17), Art. 92, α, β, γ are given in terms of λ, μ, ν. If we assume

$$\lambda = k \operatorname{cosec} \theta, \quad \mu = \sqrt{k^2 \cos^2 \phi + h^2 \sin^2 \phi}, \quad \nu = h \sin \psi, \quad h = \kappa k$$

and $\kappa^2 + \kappa'^2 = 1$, we get from (17)

$$d\alpha = \frac{-kd\theta}{\sqrt{(k^2 - h^2 \sin^2 \theta)}} = -\frac{d\theta}{\sqrt{(1 - \kappa^2 \sin^2 \theta)}},$$

$$d\beta = \frac{-kd\phi}{\sqrt{\{k^2 - (k^2 - h^2) \sin^2 \phi\}}} = -\frac{d\phi}{\sqrt{(1 - \kappa'^2 \sin^2 \phi)}},$$

$$d\gamma = \frac{kd\psi}{\sqrt{(k^2 - h^2 \sin^2 \psi)}} = \frac{d\psi}{\sqrt{(1 - \kappa^2 \sin^2 \psi)}}.$$

When $\lambda = k$, the corresponding value of θ is $\frac{\pi}{2}$; this is also the value of ϕ when $\mu = h$; and $\psi = 0$ when $\nu = 0$; hence from (17) we have

$$\alpha = F(\kappa) - F(\kappa, \theta), \;\; \beta = F(\kappa') - F(\kappa', \phi), \;\; \gamma = F(\kappa, \psi). \quad (28)$$

Also λ, μ, ν are expressed in terms of θ, ϕ, ψ by the equations

$$\lambda = k \operatorname{cosec} \theta, \quad \mu = k\Delta(\kappa', \phi), \quad \nu = h \sin \psi. \qquad (29)$$

98. **Surfaces of Revolution.**—In a confocal system, if h is zero, k remaining finite, the ellipsoids and one-sheeted hyperboloids of the system become surfaces of revolution, the form of the ellipsoids being oblate or planetary, whilst the hyperboloids of two sheets become pairs of planes, the equation of any pair being

$$\frac{x^2}{\nu^2} - \frac{y^2}{h^2 - \nu^2} = 0.$$

In accordance with the assumptions of Art. 97, we should then have

$$a = \frac{\pi}{2} - \theta, \quad \beta = \int_{\phi}^{\frac{\pi}{2}} \log \frac{1 + \tan \frac{1}{2}\phi}{1 - \tan \frac{1}{2}\phi}, \quad \gamma = \psi.$$

Here, however, it is better to proceed in a somewhat different manner, as we are thus enabled to find equations which are useful when it is required to expand the potential in a series of spherical harmonics.

It is plain that in this case we may assume two variables, η and ε, such that the semiaxes of a generating ellipse of the system of ellipsoids are denoted by $k \cosh \eta$ and $k \sinh \eta$, and the real semiaxes of a confocal hyperbola by $k \sin \varepsilon$ and $k \cos \varepsilon$; also we may put $\nu = h \cos \chi$. Thus, when the ellipsoids of the system are oblate surfaces of revolution, we have

$$\lambda = k \cosh \eta, \quad \mu = k \sin \varepsilon, \quad \nu = h \cos \chi = 0, \tag{30}$$

then the equations representing the quadrics of the confocal system become

$$\left.\begin{aligned} \frac{x^2 + y^2}{\cosh^2 \eta} + \frac{z^2}{\sinh^2 \eta} &= k^2 \\ \frac{x^2 + y^2}{\sin^2 \varepsilon} - \frac{z^2}{\cos^2 \varepsilon} &= k^2 \\ \frac{x^2}{\cos^2 \chi} - \frac{y^2}{\sin^2 \chi} &= 0 \end{aligned}\right\}. \tag{31}$$

From the first two of these equations we find that

$$x^2 + y^2 = k^2 \cosh^2 \eta \sin^2 \varepsilon, \qquad z^2 = k^2 \sinh^2 \eta \cos^2 \varepsilon;$$

hence, by the aid of the third, we obtain

$$\left.\begin{aligned} x &= k \cosh \eta \sin \varepsilon \cos \chi \\ y &= k \cosh \eta \sin \varepsilon \sin \chi \\ z &= k \sinh \eta \cos \varepsilon \end{aligned}\right\}. \tag{32}$$

Again, from (17) we have

$$a = \int_0^\eta \frac{d\eta}{\cosh \eta}, \quad \beta = \int_0^\varepsilon \frac{d\varepsilon}{\sin \varepsilon}, \quad \gamma = \frac{\pi}{2} - \chi; \tag{33}$$

whence

$$\frac{d}{d\alpha} = \cosh \eta \frac{d}{d\eta}, \quad \frac{d}{d\beta} = \sin \varepsilon \frac{d}{d\varepsilon}, \quad \frac{d}{d\gamma} = -\frac{d}{d\chi};$$

and as

$$D_1^2 = k^2 \sin^2 \varepsilon, \quad D_2^2 = k^2 \cosh^2 \eta, \quad D_3^2 = k^2 (\cosh^2 \eta - \sin^2 \varepsilon),$$

(20) becomes

$$\left\{\sin^2 \varepsilon \left(\cosh \eta \frac{d}{d\eta}\right)^2 + \cosh^2 \eta \left(\sin \varepsilon \frac{d}{d\varepsilon}\right)^2 + (\cosh^2 \eta - \sin^2 \varepsilon) \frac{d^2}{d\chi^2}\right\} V = 0. \quad (34)$$

By assuming $\sinh \eta = \zeta$, $\cos \varepsilon = \xi$, this equation becomes

$$\frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{dV}{d\zeta}\right\} + \frac{d}{d\xi}\left\{(1-\xi^2)\frac{dV}{d\xi}\right\} + \frac{\zeta^2+\xi^2}{(1+\zeta^2)(1-\xi^2)}\frac{d^2 V}{d\chi^2} = 0. \quad (35)$$

When $h = k$, the ellipsoids of the confocal system are prolate, and we may assume

$$\lambda = k \cosh \eta = k\zeta, \quad \mu^2 = k^2 \cos^2 \chi + h^2 \sin^2 \chi, \quad \nu = k \cos \varepsilon = k\xi, \quad (36)$$

then the equations representing the quadrics of the confocal system are

$$\left.\begin{aligned} \frac{x^2}{\cosh^2 \eta} + \frac{y^2 + z^2}{\sinh^2 \eta} &= k^2 \\ \frac{y^2}{\cos^2 \chi} - \frac{z^2}{\sin^2 \chi} &= 0 \\ \frac{x^2}{\cos^2 \varepsilon} - \frac{y^2 + z^2}{\sin^2 \varepsilon} &= k^2 \end{aligned}\right\}. \quad (37)$$

From these we obtain

$$\left.\begin{aligned} x &= k \cosh \eta \cos \varepsilon \\ y &= k \sinh \eta \sin \varepsilon \cos \chi \\ z &= k \sinh \eta \sin \varepsilon \sin \chi \end{aligned}\right\}. \quad (38)$$

Also from (17) we have

$$\alpha = \int_0^\eta \frac{d\eta}{\sinh \eta}, \quad \beta = -\int_{\frac{\pi}{2}}^{\chi} d\chi, \quad \gamma = -\int_{\frac{\pi}{2}}^{\epsilon} \frac{d\epsilon}{\sin \epsilon}; \tag{39}$$

whence

$$\frac{d}{d\alpha} = \sinh \eta \frac{d}{d\eta}, \quad \frac{d}{d\beta} = -\frac{d}{d\chi}, \quad \frac{d}{d\gamma} = -\sin \epsilon \frac{d}{d\epsilon};$$

also,

$$D_1^2 = k^2 \sin^2 \epsilon, \quad D_2^2 = k^2 (\cosh^2 \eta - \cos^2 \epsilon), \quad D_3^2 = k^2 \sinh^2 \eta,$$

and therefore (20) becomes in this case

$$\left\{\sin^2 \epsilon \left(\sinh \eta \frac{d}{d\eta}\right)^2 + (\cosh^2 \eta - \cos^2 \epsilon) \frac{d^2}{d\chi^2} + \sinh^2 \eta \left(\sin \epsilon \frac{d}{d\epsilon}\right)^2\right\} V = 0. \tag{40}$$

In terms of the variables ζ and ξ this equation may be written

$$\frac{d}{d\zeta}\left\{(\zeta^2 - 1)\frac{dV}{d\zeta}\right\} + \frac{d}{d\xi}\left\{(1 - \xi^2)\frac{dV}{d\xi}\right\} + \frac{\zeta^2 - \xi^2}{(\zeta^2 - 1)(1 - \xi^2)} \frac{d^2 V}{d\chi^2} = 0. \tag{41}$$

If we put $\zeta' \sqrt{-1}$ for ζ in (41) we get an equation in ζ' and ξ whose form is identical with that of (35).

It is easy to express in terms of the variables ζ, ξ, χ, the components of the force, at an external point, due to a solid ellipsoid of revolution.

In the case of an oblate ellipsoid, if X, Y, and Z denote the components required, by equations (28), Art. 24, and Art. 84, we have

$$\left.\begin{aligned} X &= \frac{3Mx}{2k^3}\left\{\tan^{-1}\frac{k}{c'} - \frac{kc'}{k^2 + c'^2}\right\} \\ Z &= \frac{3Mz}{k^3}\left\{\frac{k}{c'} - \tan^{-1}\frac{k}{c'}\right\} \end{aligned}\right\}, \tag{42}$$

where c' is the least axis of the confocal ellipsoid passing through the point x, y, z; but $c' = k\zeta$, and

$$x = k\sqrt{(1+\zeta^2)(1-\xi^2)}\cos\chi, \; y = k\sqrt{(1+\zeta^2)(1-\xi^2)}\sin\chi, \; z = k\zeta\xi.$$

and, therefore, we have

$$\left.\begin{aligned}
X &= \frac{3M}{2k^2}\sqrt{(1+\zeta^2)(1-\xi^2)}\left\{\tan^{-1}\frac{1}{\zeta} - \frac{\zeta}{1+\zeta^2}\right\}\cos\chi \\
Y &= \frac{3M}{2k^2}\sqrt{(1+\zeta^2)(1-\xi^2)}\left\{\tan^{-1}\frac{1}{\zeta} - \frac{\zeta}{1+\zeta^2}\right\}\sin\chi \\
Z &= \frac{3M}{k^2}\zeta\xi\left\{\frac{1}{\zeta} - \tan^{-1}\frac{1}{\zeta}\right\}
\end{aligned}\right\}. \quad (43)$$

For a prolate ellipsoid of revolution, we have

$$c' = k\sqrt{(\zeta^2 - 1)}, \quad x = k\zeta\xi, \quad y = k\sqrt{(\zeta^2-1)(1-\xi^2)}\cos\chi,$$
$$z = k\sqrt{(\zeta^2-1)(1-\xi^2)}\sin\chi,$$

and therefore, from equations (29), Art. 24, we obtain

$$\left.\begin{aligned}
X &= \frac{3M}{k^2}\zeta\xi\left\{\tfrac{1}{2}\log\left(\frac{\zeta+1}{\zeta-1}\right) - \frac{1}{\zeta}\right\} \\
Y &= \frac{3M}{2k^2}\sqrt{(\zeta^2-1)(1-\xi^2)}\left\{\frac{\zeta}{\zeta^2-1} - \tfrac{1}{2}\log\left(\frac{\zeta+1}{\zeta-1}\right)\right\}\cos\chi \\
Z &= \frac{3M}{2k^2}\sqrt{(\zeta^2-1)(1-\xi^2)}\left\{\frac{\zeta}{\zeta^2-1} - \tfrac{1}{2}\log\left(\frac{\zeta+1}{\zeta-1}\right)\right\}\sin\chi
\end{aligned}\right\}. \quad (44)$$

Examples.

1. Find the potential due to two confocal ellipsoidal homœoids at a point situated between them.

If a_1, b_1, c_1, a_2, b_2, c_2, be the semi-axes of the inner and outer homœoids, and M_1 and M_2 their masses, by Art. 18, and Ex. 3, Art. 75, we have

$$V = M_2\int_{a_2}^{\infty}\frac{d\lambda}{\sqrt{(\lambda^2-h^2)(\lambda^2-k^2)}} + M_1\int_{a_1}^{\infty}\frac{d\lambda}{\sqrt{(\lambda^2-h^2)(\lambda^2-k^2)}}$$
$$- M_1\int_{a_1}^{\lambda}\frac{d\lambda}{\sqrt{(\lambda^2-h^2)(\lambda^2-k^2)}}.$$

Again if
$$\alpha = \int_k^{\lambda}\frac{k\,d\lambda}{\sqrt{(\lambda^2-h^2)(\lambda^2-k^2)}}$$

it is plain that V is of the form $i\alpha + j$, where i and j are constants. If the value of V be given at the surface of each homœoid, M_1 and M_2 can be determined.

When the boundaries S_1 and S_2 of the field are the surfaces of conductors in equilibrium, the force immediately outside S_2 must be zero, and therefore, in this case, $M_2 = -M_1$, and the potential is zero at S_2.

If, however, there be another homœoidal distribution of mass on an ellipsoid enclosing the given ellipsoids S_1 and S_2, this distribution and the value of M_1 can be determined so that the potential shall have any assigned values at the surfaces S_1 and S_2.

2. If V and V' be the potentials of two confocal contrafoliated hyperboloidal homœoids at a point which has none of the surfaces of the homœoids between it and the centre, show that $V : V' = \epsilon bc : \epsilon' b' c'$, where ϵ and ϵ' denote the surface densities at the extremities of the primary axes of the homœoids, and b, c, and b', c' their secondary semi-axes.

This is an immediate consequence of Ex. 9 and 14, Art. 90, the method of proof being the same as that employed in Art 89.

3. Prove that the equipotential surfaces of a contrafoliated homœoid are the sheets of confocal hyperboloids.

This follows from Ex. 9 and 14, Art. 90. It is to be observed that the potential, though constant for each sheet, is different for the two sheets of the same hyperboloid.

4. Find the potential of a contrafoliated homœoid at a point P situated between the sheets.

This can be done by the method of Ex. 3, Art. 75, or by that of Ex. 1, Art. 91. In the present case, if ν be the primary semi-axis of a hyperboloid confocal with the homœoid, and p the central perpendicular on the tangent plane, p increases along with ν. Again, the field, in which there is a variation of the potential due to a distribution of mass equivalent to the homœoid on a confocal hyperboloid, is on the side of this hyperboloid which is next the centre. Hence, if σ be the density of this distribution,

$$\frac{dV}{dp} = 4\pi\sigma\,; \text{ and } \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

being the equation of the surface of the homœoid, we obtain

$$V = 4\pi\epsilon bc \int_0^\nu \frac{d\nu}{\sqrt{\{(h^2 - \nu^2)(k^2 - \nu^2)\}}},$$

since the potential vanishes at the centre where ν is zero.

In using this formula ν must be regarded as having the same algebraical sign as the coordinate x of the point P. The sheet for which ν is positive may be called the positive sheet, and ϵ denotes the surface density on the positive sheet at the extremity of the primary axis. At the positive sheet, and at all points in the space on the side of this sheet remote from the centre,

$$V = V_1 = 4\pi\epsilon bc \int_0^a \frac{d\nu}{\sqrt{\{(h^2 - \nu^2)(k^2 - \nu^2)\}}}.$$

At the negative sheet, and throughout the space which it separates from the centre, $V = -V_1$.

5. Find the potential due to two contrafoliated confocal homœoids at any point P.

Let the semi-axes of the homœoid next the centre be a_1, b_1, c_1, and those of

the other a_2, b_2, c_2, then, if P be situated between the two positive sheets,

$$V = 4\pi\epsilon_1 b_1 c_1 \int_0^{a_1} \frac{d\nu}{\sqrt{(h^2-\nu)(k^2-\nu^2)}} + 4\pi\epsilon_2 b_2 c_2 \int_0^{\nu} \frac{d\nu}{\sqrt{(h^2-\nu^2)(k^2-\nu^2)}}.$$

If the second integral in this equation be denoted by $\frac{\gamma}{k}$, the potential is of the form $i\gamma + j$, where i and j are constants. If the potentials at the surfaces bounding the field be given, ϵ_1 and ϵ_2 can be determined.

If P have no homœoidal surface between it and the centre,

$$V = 4\pi\,(\epsilon_1 b_1 c_1 + \epsilon_2 b_2 c_2)\frac{\gamma}{k};$$

hence for this position of P a potential containing a constant term cannot be due solely to two homœidal distributions.

When P lies between the two positive sheets, S_1 and S_2, of the homœoids, if they be the surfaces of conductors in electric equilibrium, there can be no force in the vicinity of S_1 on the side next the centre, and therefore, in this case,

$$\epsilon_1 b_1 c_1 = -\epsilon_2 b_2 c_2, \quad \text{and} \quad V = 0 \text{ at } S_1;$$

hence, if the potential be given at the surface of each of two confocal hyperboloidal conductors in electric equilibrium, it cannot be due solely to homœoidal distributions on the hyperboloids to which these surfaces belong.

6. If the equipotential surfaces of a field of force devoid of mass be confocal quadrics of the same family, find the potential at any point.

This problem has been already solved, Art 93, by the use of elliptic coordinates, but without employing this method the form of the potential can be deduced from the properties of corresponding points.

If we suppose the equipotential surfaces to be hyperboloids of one sheet, and if μ be the primary semi-axis of one of them, and p the central perpendicular on a tangent plane to this surface, the potential V is a function of μ, and

$$\frac{dV}{dp} = \frac{dV}{d\mu}\frac{d\mu}{dp} = \frac{p}{\mu}\frac{dV}{d\mu}.$$

Also, if dS and dS_1 be corresponding elements of the surfaces whose semi-axes are μ and μ_1, since dS and dS_1 are orthogonal sections of the same tube of force,

$$\frac{dV}{dp}\,dS = \left(\frac{dV}{dp}\right)_1 dS_1.$$

Hence

$$\frac{dV}{d\mu}\frac{p\,dS}{\mu} = \left(\frac{dV}{d\mu}\right)_1 \frac{p_1 dS_1}{\mu_1},$$

and therefore, by Ex. 7, Art. 90, we have

$$\frac{dV}{d\mu} = \left(\frac{dV}{d\mu}\right)_1 \frac{\sqrt{\{(\mu_1^2-h^2)(k^2-\mu_1^2)\}}}{\sqrt{\{(\mu^2-h^2)(k^2-\mu^2)\}}},$$

and

$$V = \sqrt{(\mu_1^2-h^2)(k^2-\mu_1^2)}\left(\frac{dV}{d\mu}\right)_1 \int \frac{d\mu}{\sqrt{(\mu^2-h^2)(k^2-\mu^2)}} + \text{constant.}$$

The potential is therefore of the form $i\beta + j$, where i and j are constants, and where

$$\beta = \int_h^{\mu} \frac{k\,d\mu}{\sqrt{(\mu^2-h^2)(k^2-\mu^2)}}.$$

99. **Confocal Paraboloids.**—If, in the equation of a system of central confocal quadrics, viz.:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - h^2} + \frac{z^2}{a^2 - k^2} = 1,$$

we substitute $x + a - \lambda$ for x, $\lambda + t$ for a, $h + t$ for h, and $k + t$ for k, we get

$$\frac{(x - \lambda)^2 + 2(t + \lambda)(x - \lambda)}{t^2 + 2\lambda t + \lambda^2} + \frac{y^2}{\lambda^2 - h^2 + 2(\lambda - h)\,t} + \frac{z^2}{\lambda^2 - k^2 + 2(\lambda - k)\,t} = 0.$$

If we now multiply by $2t$, and then make t infinite, we obtain the equation

$$4(x - \lambda) + \frac{y^2}{\lambda - h} + \frac{z^2}{\lambda - k} = 0, \tag{45}$$

which represents a system of confocal paraboloids.

If x, y, z, be given, λ is determined by the cubic

$$4(\lambda - x)(\lambda - h)(\lambda - k) - y^2(\lambda - k) - z^2(\lambda - h) = f(\lambda) = 0. \tag{46}$$

Here $f(\lambda)$ is positive when $\lambda = +\infty$, negative when $\lambda = k$, positive when $\lambda = h$, and negative when $\lambda = -\infty$. Hence there is one real root greater than k which may be called λ, one between k and h, which may be called μ, and one less than h which may be called ν.

The whole assemblage of confocal surfaces consists, therefore, of two systems of elliptic paraboloids turned in opposite directions, and of one system of hyperbolic paraboloids.

100. **Coordinates in terms of Parameters.**—If we put $2\eta = y$, $2\zeta = z$, in equation (46), whose roots are λ, μ, ν, we obtain

$$\lambda + \mu + \nu = x + h + k,$$

$$\lambda\mu + \mu\nu + \nu\lambda = x\,(h + k) + hk - \eta^2 - \zeta^2,$$

$$\lambda\mu\nu = hkx - k\eta^2 - h\zeta^2;$$

from which we get

$$\left.\begin{aligned} x &= \lambda + \mu + \nu - h - k \\ \frac{y^2}{4} &= \eta^2 = \frac{(\lambda - h)(\mu - h)(h - \nu)}{k - h} \\ \frac{z^2}{4} &= \zeta^2 = \frac{(\lambda - k)(k - \mu)(k - \nu)}{k - h} \end{aligned}\right\}. \tag{47}$$

Equations (47) determine the coordinates of a point in terms of the distances from the origin of the vertices of the three confocal paraboloids passing through the point. These distances, λ, μ, ν, may be called the *parameters of the paraboloids.* They vary along with the coordinates of the point through which the paraboloids pass. The quantities h and k are the *parameters of the whole system*, and are constants.

101. Laplace's Equation in Parabolic Coordinates.—In the expressions for ds_1, ds_2, ds_3 given in Art. 92, if we put $t + \lambda$, $t + \mu$, $t + \nu$, $t + h$, and $t + k$ for λ, μ, ν, h, and k, respectively, and then make t infinite, we get

$$\left.\begin{aligned} ds_1 &= \sqrt{\left(\frac{(\lambda - \mu)(\lambda - \nu)}{(\lambda - h)(\lambda - k)}\right)}\, d\lambda \\ ds_2 &= \sqrt{\left(\frac{(\lambda - \mu)(\mu - \nu)}{(\mu - h)(k - \mu)}\right)}\, d\mu \\ ds_3 &= \sqrt{\left(\frac{(\lambda - \nu)(\mu - \nu)}{(h - \nu)(k - \nu)}\right)}\, d\nu \end{aligned}\right\}. \tag{48}$$

Hence, if we assume

$$\left.\begin{aligned} \alpha &= \int_k^\lambda \frac{d\lambda}{\sqrt{\{(\lambda - h)(\lambda - k)\}}} \\ \beta &= \int_h^\mu \frac{d\mu}{\sqrt{\{(\mu - h)(k - \mu)\}}} \\ \gamma &= \int_\nu^h \frac{d\nu}{\sqrt{\{(h - \nu)(k - \nu)\}}} \end{aligned}\right\}, \tag{49}$$

$$D_1 = \sqrt{(\mu - \nu)}, \quad D_2 = \sqrt{(\lambda - \nu)}, \quad D_3 = \sqrt{(\lambda - \mu)}, \tag{50}$$

we have

$$ds_1 = D_2 D_3 d\alpha, \quad ds_2 = D_3 D_1 d\beta, \quad ds_3 = -D_1 D_2 d\gamma, \qquad (51)$$

where D_1 is independent of α, D_2 of β, and D_3 of γ.

Hence, again, proceeding as in Art 92, we get

$$\begin{aligned}\nabla^2 V &= \frac{1}{D_1^2 D_2^2 D_3^2}\left\{D_1^2 \frac{d^2 V}{d\alpha^2} + D_2^2 \frac{d^2 V}{d\beta^2} + D_3^2 \frac{d^2 V}{d\gamma^2}\right\} \\ &= \frac{1}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}\left\{(\mu-\nu)\frac{d^2 V}{d\alpha^2} + (\lambda-\nu)\frac{d^2 V}{d\beta^2}\right. \\ &\qquad\qquad \left. + (\lambda-\mu)\frac{d^2 V}{d\gamma^2}\right\}, \qquad (52)\end{aligned}$$

and Laplace's Equation becomes

$$(\mu-\nu)\frac{d^2 V}{d\alpha^2} + (\lambda-\nu)\frac{d^2 V}{d\beta^2} + (\lambda-\mu)\frac{d^2 V}{d\gamma^2} = 0. \qquad (53)$$

102. **Field of Force.**—If the equipotential surfaces of a field of force, devoid of mass, be confocal paraboloids of the same family, V must be a function of one of the parameters α, β, γ. If it be a function of α, we have then, as in Art. 93, $V = i\alpha + j$, and a similar result holds good if V be a function of β or of γ.

It is now easy, as in Arts. 94, 95, 96, to obtain the potential at any point in a field of force when the equipotential surfaces are paraboloids, if the potentials of the surfaces bounding the field be given.

When the paraboloids bounding the field belong to the family whose parameters are denoted by λ, and are the surfaces of conductors in equilibrium, the density σ at any point P of the boundary is proportional to $\frac{dV}{ds_1}$, that is, to

$$\frac{1}{\sqrt{(\lambda-\mu)(\lambda-\nu)}}\frac{dV}{d\alpha},$$

but from the general form of V in this case we see that $\frac{dV}{d\alpha}$ is

constant, and therefore σ varies as

$$\frac{1}{\sqrt{(\lambda-\mu)(\lambda-\nu)}},$$

where μ and ν are the parameters of the confocals passing through P.

The element ds_1 is the normal distance between the surface whose parameter is λ and the consecutive confocal surface whose parameter is $\lambda + d\lambda$, and the difference of the coordinates of the two extremities of ds_1 is dx, where dx is obtained from equations (47) by supposing λ to vary and μ and ν to remain constant. Again, if ϖ_1 be the angle which the normal at P to the surface having λ for parameter makes with the axis of x, we have $dx = ds_1 \cos\varpi_1$. Hence, by (48), we have

$$\cos\varpi_1 = \frac{dx}{ds_1} = \frac{d\lambda}{ds_1} = \sqrt{\left(\frac{(\lambda-h)(\lambda-k)}{(\lambda-\mu)(\lambda-\nu)}\right)}, \qquad (54)$$

whence the surface density σ varies as $\cos\varpi_1$.

It appears, therefore, that the distribution of mass on the surface is equivalent to a homogeneous shell comprised between two paraboloids such that one can be made to coincide with the other by a displacement of translation parallel to its axis.

Such a shell may be termed a *paraboloidal homœoid*.

103. **Relation between Parameters.**—From equations (49), λ, μ, ν can be determined in terms of α, β, γ.

We may write (49) in the form

$$\left.\begin{aligned} \alpha &= \int_k^\lambda \frac{d\lambda}{\sqrt{\{\lambda - \frac{1}{2}(k+h)\}^2 - \frac{1}{4}(k-h)^2}} \\ \beta &= \int_h^\mu \frac{d\mu}{\sqrt{\frac{1}{4}(k-h)^2 - \{\frac{1}{2}(k+h) - \mu\}^2}}, \\ \gamma &= \int_\nu^h \frac{d\nu}{\sqrt{\{\frac{1}{2}(k+h) - \nu\}^2 - \frac{1}{4}(k-h)^2}} \end{aligned}\right\}; \qquad (55)$$

whence, if we assume

$$\lambda - \tfrac{1}{2}(k+h) = \tfrac{1}{2}(k-h)\cosh\theta, \quad \tfrac{1}{2}(k+h) - \mu = \tfrac{1}{2}(k-h)\cos\phi,$$
$$\tfrac{1}{2}(k+h) - \nu = \tfrac{1}{2}(k-h)\cosh\psi,$$

we get $\alpha = \theta$, $\beta = \phi$, $\gamma = \psi$. Hence, we have

$$\left.\begin{aligned}\lambda &= \tfrac{1}{2}(k+h) + \tfrac{1}{2}(k-h)\cosh\alpha,\\ \mu &= \tfrac{1}{2}(k+h) - \tfrac{1}{2}(k-h)\cos\beta,\\ \nu &= \tfrac{1}{2}(k+h) - \tfrac{1}{2}(k-h)\cosh\gamma\end{aligned}\right\}. \tag{56}$$

104. **Corresponding Points.**—Two points whose coordinates are x, y, z and x', y', z', situated on confocal paraboloids of the same family whose parameters are λ and λ', correspond when

$$x - \lambda = x' - \lambda', \quad \frac{y}{\sqrt{(\lambda - h)}} = \frac{y'}{\sqrt{(\lambda' - h)}}, \quad \frac{z}{\sqrt{(\lambda - k)}} = \frac{z'}{\sqrt{(\lambda' - k)}}. \tag{57}$$

If P_1 and P_2 be two points on a paraboloid, and P'_1 and P'_2 be the corresponding points, then $P_1P'_2 = P_2P'_1$.

For

$$P_1P'^2_2 = (x_1 - x'_2)^2 + (y_1 - y'_2)^2 + (z_1 - z'_2)^2$$

$$= \left\{x'_1 - x_2 + 2(\lambda - \lambda')\right\}^2 + \left\{\sqrt{\left(\frac{\lambda - h}{\lambda' - h}\right)}\, y'_1 - \sqrt{\left(\frac{\lambda' - h}{\lambda - h}\right)}\, y_2\right\}^2$$
$$+ \left\{\sqrt{\left(\frac{\lambda - k}{\lambda' - k}\right)}\, z'_1 - \sqrt{\left(\frac{\lambda' - k}{\lambda - k}\right)}\, z_2\right\}^2.$$

$$= (x'_1 - x_2)^2 + (y'_1 - y_2)^2 + (z'_1 - z_2)^2$$
$$+ (\lambda - \lambda')\left\{\frac{y'^2_1}{\lambda' - h} + \frac{z'^2_1}{\lambda' - k} - \frac{y_2^2}{\lambda - h} - \frac{z_2^2}{\lambda - k}\right\}$$
$$+ 4(\lambda - \lambda')(x'_1 - x_2 + \lambda - \lambda') = P'_1P_2^2.$$

By supposing λ to vary in equations (47), whilst μ and ν remain constant, it appears that the line of intersection of the surfaces whose parameters are μ and ν meets confocal paraboloids of the remaining family in corresponding points.

Hence, when the equipotential surfaces of a field of force are confocal paraboloids of the same family, corresponding points lie on the same line of force.

105. **Paraboloids of Revolution.**—If $h = k$, the two systems of elliptic paraboloids become systems of paraboloids

of revolution turned in opposite directions, whilst the hyperbolic paraboloids become planes which may be represented by the equation

$$\frac{y^2}{\sin^2\phi} - \frac{z^2}{\cos^2\phi} = 0. \tag{58}$$

In this case μ, one root of equation (46), is equal to h. The remaining roots λ and ν are determined by a quadratic equation, from which we obtain

$$\lambda + \nu = h + x, \quad \lambda\nu = hx - \frac{y^2 + z^2}{4}. \tag{59}$$

Hence, we have

$$x = \lambda + \nu - h, \quad \frac{y^2 + z^2}{4} = (\lambda - h)(h - \nu). \tag{60}$$

The assumptions by which the equations were obtained which connect λ and ν with α and γ are, in this case, not legitimate; but if we assume

$$\alpha = \int_{2h}^{\lambda} \frac{d\lambda}{\lambda - h}, \qquad \gamma = -\int_0^{\nu} \frac{d\nu}{h - \nu}, \tag{61}$$

the expressions for ds_1 and ds_3 are found as in Art. 101, and are given by equations (51) when in those equations h is substituted for μ. Again it is plain that here $ds_2 = \sqrt{(y^2 + z^2)}\, d\phi$, and therefore by (60), we have $ds_2 = 2\sqrt{(\lambda - h)(h - \nu)}\, d\phi$. In this case, accordingly, μ in equations (52) and (53) is to be replaced by h, and $\frac{d^2}{d\beta^2}$ by $\frac{1}{4}\frac{d^2}{d\phi^2}$.

From equations (61) we have

$$\alpha = \log\frac{\lambda - h}{h}, \qquad \gamma = \log\frac{h - \nu}{h};$$

whence $$\lambda = h(1 + e^{\alpha}), \qquad \nu = h(1 - e^{\gamma}). \tag{62}$$

Since $\lambda - h$ and λ are the distances of the vertex of the paraboloid whose parameter is λ from the focus and from the origin, h is the distance of the focus from the origin. Hence, if we take the focus as origin, from equations (58), (60), and (62), we obtain

$$x = h(e^{\alpha} - e^{\gamma}), \qquad y^2 = 4h^2 e^{\alpha+\gamma}\sin^2\phi, \qquad z^2 = 4h^2 e^{\alpha+\gamma}\cos^2\phi. \tag{63}$$

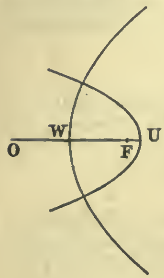
Transforming to polar coordinates whose meridians intersect in the axis of x we get

$$\frac{r\cos\theta}{h} = e^{\alpha} - e^{\gamma}, \qquad \frac{r\sin\theta}{h} = 2e^{\frac{\alpha+\gamma}{2}}.$$

If we solve the equations connecting e^{α} and e^{γ} with $\frac{r\sin\theta}{h}$ and $\frac{r\cos\theta}{h}$, we get

$$e^{\alpha} = \frac{r(1+\cos\theta)}{2h}, \qquad e^{\gamma} = \frac{r(1-\cos\theta)}{2h},$$

and finally we obtain



$$\left.\begin{aligned} \frac{\alpha}{2} &= \log\left(\frac{r^{\frac{1}{2}}\cos\frac{1}{2}\theta}{h^{\frac{1}{2}}}\right), \\ \frac{\beta}{2} &= \phi, \\ \frac{\gamma}{2} &= \log\left(\frac{r^{\frac{1}{2}}\sin\frac{1}{2}\theta}{h^{\frac{1}{2}}}\right) \end{aligned}\right\}. \quad (64)$$

where ϕ is measured from the axis of z.

The figure represents a section of two paraboloids of different families, O being the origin, and F the focus; then

$$OF = h, \quad OU = \lambda, \quad OW = \nu.$$

Examples.

1. If a thick homogeneous shell be comprised between two coaxal elliptic paraboloids, such that one can be made to coincide with the other by a displacement of translation parallel to their axis, prove that the shell exercises no attraction at any point in the interior space.

Let $x + \lambda = \frac{y^2}{p} + \frac{z^2}{q}$ be the equation of a paraboloid, then, if x', y', z', x'', y'', z'' be the coordinates of two points P and Q on it, we have, by subtraction,

$$x' - x'' = (y' - y'')\frac{y' + y''}{p} + (z' - z'')\frac{z' + z''}{q}$$

Hence if x, y, z, be the coordinates of M the middle point of PQ, and if PQ be parallel to the line whose equations are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}, \quad \text{we have} \quad l = \frac{2my}{p} + \frac{2nz}{q}.$$

Hence the locus of M for all chords parallel to a fixed line is a plane parallel to the axis, and this locus is independent of λ. If now we have two paraboloids whose equations differ only in the value of λ, and a chord PQ of one meet the other in the points $P'Q'$, since the middle point of PQ coincides with that of $P'Q'$, we must have $P'P = QQ'$. The required result is now apparent by the method of Art. 18.

A shell such as that which is the subject of the present Example may be called a *thick homœoid,* and if infinitely thin, simply a *homœoid.*

2. If a distribution of mass on the surface of a paraboloid be equivalent to a homœoid, prove that the density at any point varies as cos ϖ, where ϖ is the angle which the normal at the point makes with the axis of the paraboloid.

3. If dS and dS' be corresponding elements of confocal paraboloids of the same family whose parameters are λ and λ', and if σ and σ' be the surface densities at dS and dS' of homœoidal distributions of mass, prove that

$$\sigma dS : \sigma' dS' = \epsilon \sqrt{(\lambda - h)(\lambda - k)} : \epsilon' \sqrt{(\lambda' - h)(\lambda' - k)},$$

where ϵ and ϵ' are the surface densities at the vertices of the paraboloids.

Since σ varies as cos ϖ, we have $\sigma = \epsilon \cos \varpi$, and therefore,

$$\sigma dS = \epsilon dS \cos \varpi = \epsilon\, dydz.$$

Hence, by (57), Art. 104, we obtain the required result.

4. If V and V' be the potentials of two confocal paraboloidal homœoids of the same family whose parameters are λ and λ', and vertex surface densities ϵ and ϵ', and if P and P' be corresponding points on these homœoids, prove that

$$V_{P'} : V'_{P} :: \epsilon \sqrt{(\lambda - h)(\lambda - k)} : \epsilon' \sqrt{(\lambda' - h)(\lambda' - k)}.$$

This follows from the last Example by means of Art 104.

5. If V and V' be the potentials at any external point of two confocal elliptic paraboloidal homœoids of the same family whose parameters are λ and λ', prove that

$$V : V' = \epsilon \sqrt{(\lambda - h)(\lambda - k)} : \epsilon' \sqrt{(\lambda' - h)(\lambda' - k)}.$$

where ϵ and ϵ' are the surface densities at the vertices.

This follows from Examples 1 and 4.

6. Show that the equipotential surfaces of an elliptic paraboloidal homœoid are confocal paraboloids of the same family.

7. Find the potential of an elliptic paraboloidal homœoid at any external point.

If ds be the element of a line of force at the surface S of a confocal paraboloid whose parameter is λ, we have

$$\frac{dV}{ds} = \frac{dV}{d\lambda}\frac{d\lambda}{ds},$$

but at the vertex $d\lambda = ds$, and, therefore,

$$\frac{dV}{d\lambda} = -4\pi\epsilon,$$

where ϵ denotes the vertex surface density of a distribution of mass on S equivalent to the homœoid; also, if λ_1 and ϵ_1 be the parameter and vertex surface density of the given homœoid,

$$\epsilon\sqrt{(\lambda - h)(\lambda - k)} = \epsilon_1\sqrt{(\lambda_1 - h)(\lambda_1 - k)}.$$

Hence

$$V = \text{constant} - 4\pi\epsilon_1\sqrt{(\lambda_1 - h)(\lambda_1 - k)}\int\frac{d\lambda}{\sqrt{(\lambda - h)(\lambda - k)}}.$$

The constant in this equation has an *infinite value*, as appears most readily by considering a homœoid of revolution and the value of the potential at its focus, which is the same as that at its surface.

8. Find the potential due to two confocal elliptic paraboloidal homœoids of the same family at any point between them when

$$\epsilon_2\sqrt{(\lambda_2 - h)(\lambda_2 - k)} = -\epsilon_1\sqrt{(\lambda_1 - h)(\lambda_1 - k)},$$

where ϵ_1 and ϵ_2 are the surface densities at the vertices, and λ_1 and λ_2 the parameters of the paraboloids.

In this case the potential is zero, at the outer paraboloid, and at any point between the surfaces, we have

$$V = 4\pi\epsilon_1\sqrt{(\lambda_1 - h)(\lambda_1 - k)}\int_\lambda^{\lambda_2}\frac{d\lambda}{\sqrt{(\lambda - h)(\lambda - k)}}.$$

9. If the equipotential surfaces of a field of force devoid of mass be confocal paraboloids of the same family, find the potential at any point.

If V be the potential at any point P of the field, dS the element of the equipotential surface passing through P whose parameter is μ, and dS_1 the corresponding element of any other equipotential surface whose parameter is μ_1, and if ds and ds_1 denote the elements of lines of force perpendicular to dS and dS_1 and making angles ϖ and ϖ_1 with the paraboloidal axis; then, by (47), Art. 100, and (3), Art. 28, we have

$$\frac{dV}{d\mu}\,dydz = \frac{dV}{d\mu}\,dS\cos\varpi = \frac{dV}{d\mu}\frac{d\mu}{ds}\,dS = \frac{dV}{ds}\,dS = \left(\frac{dV}{ds}\right)_1 dS_1 = \left(\frac{dV}{d\mu}\right)_1 dy_1dz_1.$$

Hence, by (57), Art. 104, we have

$$V = \left(\frac{dV}{d\mu}\right)_1\sqrt{(\mu_1 - h)(k - \mu_1)}\int\frac{d\mu}{\sqrt{(\mu - h)(k - \mu)}} + \text{constant}.$$

If

$$\beta = \int_h\frac{d\mu}{\sqrt{(\mu - h)(k - \mu)}},$$

the potential V is therefore of the form $i\beta + j$, where i and j are constants.

10. If the equipotential surfaces of a field of force be paraboloids of revolution, show that the force component at any point P in the direction joining P to the focus F varies inversely as PF.

This follows from the expression for the potential, since

$$(\lambda - h) \sec^2 \varpi = PF = \lambda - \nu.$$

11. If the boundaries of the field in the last Example be the surfaces of conductors in electric equilibrium, show that the density of the distribution of mass on one of these surfaces varies inversely as the focal perpendicular on the tangent plane.

106. **Uniplanar Distribution.**—Results analogous to those obtained in Arts. 99–105 hold good for a uniplanar distribution of mass acting inversely as the distance.

The equation

$$4(x - \lambda) + \frac{y^2}{\lambda - h} = 0 \tag{65}$$

represents a system of confocal parabolas containing two families such that the curves belonging to the one and those belonging to the other are turned in opposite directions.

In fact if x and y be given in (65), the quadratic equation for determining λ has two roots, one between $+\infty$ and h which may be called λ, and one between h and $-\infty$ which may be called μ. Then we have

$$x = \lambda + \mu - h, \quad y^2 = 4(\lambda - h)(h - \mu). \tag{66}$$

Corresponding points whose coordinates are x, y, and x' y', on confocal parabolas of the same family, whose parameters are λ and λ', are defined by the equations

$$x - \lambda = x' - \lambda', \quad \frac{y}{\sqrt{\lambda - h}} = \frac{y'}{\sqrt{\lambda' - h}}. \tag{67}$$

If ds and dt denote the elements of the curves whose parameters are λ and μ at the point x, y, since these curves cut orthogonally, by (66) we have

$$\frac{d\lambda}{dt} = \frac{dx}{dt} = \cos \varpi,$$

where ϖ is the angle which the normal to ds makes with the axis of the parabolas.

If the equipotential curves of a field of force devoid of mass be confocal parabolas, and we denote by λ the parameter of the parabola passing through the point x, y, we have

$$\frac{dV}{d\lambda}\,dy = \frac{dV}{d\lambda}\,ds\cos\varpi = \frac{dV}{dt}\,ds = \left(\frac{dV}{dt}\right)_1 ds_1 = \left(\frac{dV}{d\lambda}\right)_1 dy_1,$$

when y and y_1 are the coordinates of corresponding points on the parabolas whose parameters are λ and λ_1.

Hence

$$V = \left(\frac{dV}{d\lambda}\right)_1 \sqrt{\lambda_1 - h}\int\frac{d\lambda}{\sqrt{\lambda - h}} + \text{constant.} \tag{68}$$

If we assume

$$a = \int_h^\lambda \frac{d\lambda}{2\sqrt{h(\lambda - h)}} = \sqrt{\left(\frac{\lambda - h}{h}\right)}, \tag{69}$$

V is of the form $ia + j$, where i and j are constants.

If the parameter of an equipotential curve be μ, it can be shown in like manner that the potential V is of the form $i\beta + j$, where

$$\beta = \sqrt{\left(\frac{h - \mu}{h}\right)}. \tag{70}$$

The parameters λ and μ are expressed in terms of the parameters a and β by the equations

$$\lambda = h(1 + a^2), \quad \mu = h(1 - \beta^2). \tag{71}$$

If we transform to the focus as origin the coordinates of the point P of intersection of the parabolas whose parameters are λ and μ, we obtain

$$x = \lambda + \mu - 2h = h(a^2 - \beta^2), \qquad y^2 = 4h^2a^2\beta^2.$$

Hence, if the distance of P from the focus be denoted by r, and the angle it makes with the axis of the parabolas by θ, we get $h(a^2 + \beta^2) = r$, and we find

$$a = \left(\frac{r}{h}\right)^{\frac{1}{2}}\cos\tfrac{1}{2}\theta, \qquad \beta = \left(\frac{r}{h}\right)^{\frac{1}{2}}\sin\tfrac{1}{2}\theta. \tag{72}$$

From (72) we have

$$\alpha + \beta\sqrt{-1} = h^{-\frac{1}{2}}(x + y\sqrt{-1})^{\frac{1}{2}}. \tag{73}$$

Hence α and β are conjugate functions of x and y. (See Art. 53).

When the equipotential curves due to a uniplanar distribution of mass are *central* confocal conics of the same family, we may employ the method of Ex. 2, Art. 53.

It is easy to see otherwise, as in Art. 92, that

$$ds = \sqrt{\frac{\lambda^2 - \mu^2}{h^2 - \mu^2}}\, d\mu, \quad dt = \sqrt{\frac{\lambda^2 - \mu^2}{\lambda^2 - h^2}}\, d\lambda, \tag{74}$$

where ds and dt are the arc elements of the ellipse and hyperbola which pass through the same point, and whose major axes are 2λ and 2μ respectively.

Hence, if we put $\lambda = h \cosh \eta$, $\mu = h \cos \xi$, we obtain

$$ds = -\sqrt{\lambda^2 - \mu^2}\, d\xi, \quad dt = \sqrt{\lambda^2 - \mu^2}\, d\eta. \tag{75}$$

From these equations we get

$$\nabla^2 V = \frac{1}{\lambda^2 - \mu^2}\left(\frac{d^2 V}{d\xi^2} + \frac{d^2 V}{d\eta^2}\right). \tag{76}$$

In the case of confocal *parabolas* we find, as in Art. 101, that

$$\left.\begin{aligned} ds &= \sqrt{\frac{\lambda - \mu}{h - \mu}}\, d\mu = -2\sqrt{h(\lambda - \mu)}\, d\beta \\ dt &= \sqrt{\frac{\lambda - \mu}{\lambda - h}}\, d\lambda = 2\sqrt{h(\lambda - \mu)}\, d\alpha \end{aligned}\right\}, \tag{77}$$

where α and β are given by equations (69) and (70).

From (77) we obtain

$$\nabla^2 V = \frac{1}{4h(\lambda - \mu)}\left(\frac{d^2 V}{d\alpha^2} + \frac{d^2 V}{d\beta^2}\right). \tag{78}$$

CHAPTER VI.

ELECTRIC IMAGES.

107. **Conductor put to Earth.**—When a conductor C in connexion with the ground is in electric equilibrium, its potential is the same as that of the surface of the earth. If the connexion be made by means of a thin wire, and if the distance of C from the earth's surface be large compared with the dimensions and mutual distances of C and the other conductors in the vicinity, the potential of the electricity distributed on the entire system of conductors is approximately zero at all points of C. Hence we may assume that a conductor in connexion with the ground, when in a state of electric equilibrium, is at potential zero.

Such a conductor is said to be *put to earth.*

108. **Electrified Point.**—If a body A charged with electricity be brought into the presence of a conductor C which is put to earth, the potential of the electric mass on A is not zero at C, and in order that the total potential at C should be zero there must be a separation and distribution of electric mass on C, such that the potential at C due to this distribution along with that on A is zero. This distribution on C is said to be *induced* by A.

If A be a conductor the distribution induced on C disturbs the previously existing constancy of the potential at A, and the problem presented for solution is to find the distribution of a given charge on A, and the charge and distribution on C, so that the potential due to the two distributions shall be constant at the surface of A and zero at the surface of C.

In order to simplify the problem we may, for mathematical purposes, suppose the body A to shrink to a point, whilst the electric mass which it contains remains finite. This hypothesis cannot be realized in nature, but is analogous

to the hypothesis in Dynamics of the existence of separate particles of finite mass, and assists in the solution of problems whose conditions *may* exist in experience.

If we attend to what has been said above we may use the expression *electrified point* and may specify the electric mass or charge there concentrated.

109. **Image of Electrified Point.**—If a conductor C, put to earth, be in the presence of a point A, at which a charge e of electricity is concentrated, this charge induces, as we have seen, a certain distribution of electricity on the surface of C. If this surface S be closed, or divide space into two regions, we may imagine a distribution e' of mass in the region $\mathfrak{S}'$ which S separates from A, whose potential at each point of S is the same as that of the actual distribution on S. Then, for the whole of the region $\mathfrak{S}$ on the same side of S as A, the potential of e' is the same as that of the distribution on S.

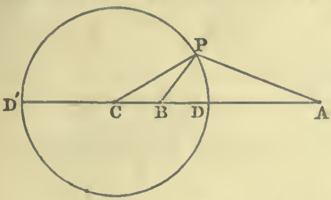
Accordingly, as the distribution e' is a hypothetical distribution of mass producing the same effect in the region containing e as the actual modification of the electric condition of the surface S produced by e, the distribution e' may be called *the image of* e *in the surface* S.

When e' is concentrated at a point B, this point may be called *the image of* A *in* S, and we have the definition: Two points A and B on opposite sides of a surface S, are *images* of each other in that surface if they be such that a given charge e at A, and a corresponding charge e' at B, produce a potential which is zero at each point of S.

A surface S in which the image of a point A is another point B must be either a plane or a sphere; for, if r and r' be the distances of any point P on S from A and B, since the potential at P is zero, we have $\frac{e}{r} + \frac{e'}{r'} = 0$, which is the equation of a plane or of a sphere according as $e' = -e$ or otherwise, except e' and e have the same algebraical sign, in which case S is altogether at an infinite distance from A and B.

110. **Image of Point in Sphere.**—If B be the image of A in the sphere S whose centre is C, the resultant force

due to e at A and to e' at B must be along the normal at every point of S, but at the point D in which CA meets S, the force due to e is normal; so also, therefore, is that due to e', and consequently B must lie in the line CA. Again, if AC produced meet S in D', we have



$$\frac{e}{AD} + \frac{e'}{BD} = 0 = \frac{e}{AD'} + \frac{e'}{BD'},$$

whence $D'D$ is cut harmonically in B and A, and therefore $CA.CB = CD^2$. Accordingly, B is a point in CA determined by this equation if A and B be images of each other in S.

Let P be any point of the sphere S, and let $CA = f$, $CB = f'$, $CP = a$; then, from the similar triangles ACP and PCB, we have $AP : BP = f : a$. Hence, if we assume

$$e' = -\frac{ea}{f}, \qquad \text{we obtain} \qquad \frac{e}{AP} + \frac{e'}{BP} = 0,$$

and B *is* the image of A, and the charge e' at B is given by the equation

$$e' = -\frac{ea}{f} = -e\sqrt{\frac{f'}{f}}. \tag{1}$$

111. **Induced Distribution on Sphere.**—The distribution of electricity on a sphere at potential zero under the influence of an external electrified point A can now be readily determined.

Adopting the same notation as that of the preceding Article, let σ denote the density of the surface distribution at any point P of the sphere whose distances from A and B are r and r'. Then, if R be the resultant force at P, this force is in the direction of the normal; and therefore, if we resolve the forces due to e at A, and to e' at B along CP and any other direction, the resultant along this latter direction is zero, and R is the sum of the components along CP. Hence, resolving along CP and CA, if the direction of R be from C to P, we have

$$R = \frac{e}{r^2}\,\frac{a}{r} + \frac{e'}{r'^2}\,\frac{a}{r'};$$

but $$\frac{e'}{r'} = -\frac{e}{r}, \quad \text{and} \quad \frac{1}{r'} = \frac{f}{a}\frac{1}{r};$$

whence $$R = \frac{e(a^2 - f^2)}{ar^3};$$

then, as the potential is zero throughout the sphere, by Art. 29, we have

$$\sigma = \frac{R}{4\pi} = \frac{e}{4\pi a}\,\frac{(a^2 - f^2)}{r^3} \qquad (2)$$

Since $f > a$, it appears from (2) that R and σ are both negative, that is, if e be positive the charge on the sphere is everywhere negative, and the force at its surface tends to drive positive electricity towards the centre.

Since the potential in external space of the distribution on the sphere is the same as that due to e' at B, by Art. 37 the total charge on the sphere is e', that is, $-e\frac{a}{f}$.

If the sphere be hollow, and the point A in its interior, the potential is zero throughout external space (Art. 62), and therefore if the direction of R be from C to P as before, we have

$$\sigma = -\frac{R}{4\pi} = -\frac{e}{4\pi a}\,\frac{a^2 - f^2}{r^3}. \qquad (3)$$

Here $a > f$, and σ, as before, is negative, but the force at the inner surface of the sphere tends to drive positive electricity outwards.

In this case, since the potential in external space is zero, the total charge on the sphere is $-e$.

112. **Insulated Sphere.**—If an insulated spherical conductor S, to which a charge E of electricity has been imparted, be in the presence of an electrified point A, we may suppose the sphere to have gone through a previous process in which it is at first put to earth in the presence of A and then insulated. It has now a charge E_1 induced by the charge e at A, and the potential is zero at each point of S. Finally, we may suppose an additional charge E_2 to be

communicated to the sphere, and to be uniformly distributed over its surface; the potential is then constant at S, and provided that $E_1 + E_2 = E$, we obtain the actual distribution on the surface of the charged insulated sphere.

To prove this we have only to show that there cannot be two different functions of the coordinates, U and U', representing the potential in external space of the distribution on the insulated sphere having a given total charge E.

If there were two such functions, and if r denote the distance of A from any point on S, σ and σ' the densities, at that point, of the surface distributions corresponding to U and U', and ν the normal to S drawn outwards, we should have

$$4\pi\sigma + \frac{d}{d\nu}\left(U + \frac{e}{r}\right) = 0, \quad 4\pi\sigma' + \frac{d}{d\nu}\left(U' + \frac{e}{r}\right) = 0;$$

whence
$$\int \frac{d}{d\nu}(U - U')\, dS = 4\pi \int (\sigma' - \sigma)\, dS = 0.$$

Also at S we have

$$U + \frac{e}{r} = C, \quad U' + \frac{e}{r} = C',$$

whence
$$U - U' = C - C'.$$

Now let $U - U' = \phi$, then

$$\int \phi \frac{d\phi}{d\nu}\, dS + \int \phi \nabla^2 \phi\, d\mathfrak{S} = (C - C') \int \frac{d\phi}{d\nu}\, dS = 0,$$

where the volume integral is taken throughout the whole of space outside the sphere. Hence, by (9), Art. 58, ϕ is zero everywhere outside the sphere, and $U = U'$. Hence also, $\sigma = \sigma'$, and there is only one possible distribution on the sphere of a given total charge which satisfies the conditions imposed.

It is now easy to find σ the density at any point of the distribution on the surface of the sphere, for $\sigma = \sigma_1 + \sigma_2$, where σ_1 is the density of the distribution when the sphere is at potential zero, and $\sigma_2 = \dfrac{E_2}{4\pi a^2}$. Then, by (2) we have

$$\sigma_1 = -\frac{1}{4\pi a^2} \frac{ea(f^2 - a^2)}{r^3}; \text{ also } E_2 = E + \frac{a}{f} e;$$

whence

$$\sigma = \frac{1}{4\pi a^2}\left\{E + \frac{a}{f}e - \frac{ea(f^2 - a^2)}{r^3}\right\}. \tag{4}$$

The potential U of the distribution on the sphere at any external point Q is the same as that due to a charge E_2, or $E + \frac{a}{f}e$, at its centre C, together with a charge e', or $-\frac{a}{f}e$, at B. Hence

$$U = \left(E + \frac{a}{f}e\right)\frac{1}{CQ} - \frac{a}{f}\frac{e}{BQ}; \tag{5}$$

and if V be the total potential at Q, we have

$$V = \frac{e}{AQ} - \frac{a}{f}\frac{e}{BQ} + \left(E + \frac{a}{f}e\right)\frac{1}{CQ}. \tag{6}$$

If the value of the potential V at the surface of the sphere be denoted by L, this potential is the same as that due to the charge E_2 at the centre. Hence,

$$L = \frac{E}{a} + \frac{e}{f}, \text{ and therefore } E = La - \frac{ea}{f}. \tag{7}$$

Also, at the point P on the surface,

$$\sigma = \frac{1}{4\pi a}\left\{L - \frac{e(f^2 - a^2)}{AP^3}\right\}. \tag{8}$$

The expression for U given by equation (5) can easily be found directly. For, at any point P on the surface of the sphere S the potential U must be of the form $L - \frac{e}{AP}$, where L is constant. Hence, at this surface,

$$U = \frac{La}{CP} - \frac{a}{f}\frac{e}{BP};$$

and therefore, by Art. 64, at any point Q in external space,

$$U = \frac{La}{CQ} - \frac{a}{f}\frac{e}{BQ}.$$

This is the potential due to a mass La at C, together with a mass $-\frac{ea}{f}$ at B. Accordingly

$$E = La - \frac{ea}{f}, \quad \text{and} \quad La = E + \frac{ea}{f}.$$

113. **Sphere with Electrified Point in its Interior.**—If a charge e be at a point A in the interior of a hollow charged insulated spherical conductor, the distribution of mass on the interior surface of this conductor is given by (3), Art. 111, the total mass being $-e$, and on the external surface there is a uniform distribution of the total mass $E + e$, where E is the charge which was imparted to the conductor.

At an internal point Q, the potential V is given by the equation

$$V = \frac{E+e}{a} + \frac{e}{AQ} - \frac{ea}{f}\frac{1}{BQ}, \tag{9}$$

where B is the image of A in the sphere.

At an external point the potential is that due to a charge $E + e$ at the centre of the sphere.

114. **Sphere in Field of Uniform Force.**—If the force throughout a certain region of space be of uniform magnitude F and parallel to a fixed direction, it may be regarded as due to an infinite mass M, situated at an infinite distance R in a direction opposite to that of the force, provided

$$\frac{M}{R^2} = F.$$

If now an insulated sphere, whose centre is C, be placed in the field, a distribution of electricity is induced on its surface, whose potential U in external space can be deduced from (5) by supposing $E = 0$ in that equation. In the present case $e = M$, $f = R$, and CB is given by the equation $CB = \frac{a^2}{R}$. Hence

$$e\frac{a}{f} \cdot CB = \frac{Ma^3}{R^2} = Fa^3;$$

and, accordingly, in external space the potential due to the

distribution on the sphere is mathematically the same as that of a magnetic particle at C, whose axis is codirectional with the uniform force, and whose magnetic moment is Fa^3. Such a combination of attractive and repulsive centres of force is termed a *doublet.*

If we take C for origin, and a line in the direction of the uniform force as axis of x, we have

$$U = Fa^3 \frac{x}{r^3}; \tag{10}$$

and the total potential V in space external to the sphere is given by the equation

$$V = F\left(\frac{a^3}{r^3} - 1\right)x + K, \tag{11}$$

where K denotes a constant.

EXAMPLES.

1. If mass be distributed on the surface of a sphere so that the density at any point varies inversely as the cube of its distance from a fixed point A, show that the distribution is centrobaric, and find the baric centre.

Let f denote the distance of A from the centre C of the sphere, a its radius, and σ the density at any point P on its surface; then

$$\sigma = \frac{K}{AP^3}; \quad \text{and if we assume} \quad e = \frac{4\pi aK}{a^2 - f^2},$$

by (3) and (2), Art. 111, the distribution is identical with that induced on a sphere at potential zero by a charge $\mp e$ at the point A. Hence, if A be inside the sphere, A is the baric centre, and e represents the total mass on the sphere.

If A be outside the sphere, the image of A in its surface is the baric centre, and the total mass is $-e\dfrac{a}{f}$.

2. Find the value of the integral $\displaystyle\int \frac{f^2 - a^2}{r^3}\, dS$ taken over the surface S of a sphere whose radius is a, where r and f are the distances of dS and the centre of the sphere from an external point A. *Ans.* $\dfrac{4\pi a^2}{f}$.

3. What is the value of the integral in the last Example if A be on the sphere? *Ans.* $4\pi a$.

4. Find the distribution of mass on an uncharged insulated sphere in presence of an electrified point A.

Let e denote the charge at A, f its distance from the centre of the sphere, and σ the density at the point P on its surface; then

$$\sigma = \frac{e}{4\pi a}\left\{\frac{1}{f} - \frac{f^2 - a^2}{AP^3}\right\}.$$

5. Show that the potential at the surface of an insulated sphere having a charge E, and under the influence of a quantity e of electricity at an external point, is equal to the mean value of the potential of e taken throughout the sphere, together with the potential of E, uniformly distributed over its surface.

6. Find the force which an insulated sphere, having a charge E, exerts on a quantity e of electricity concentrated at an external point A.

If F denote the force, and r the distance of A from the centre of the sphere,

$$F = e\left\{\frac{E}{r^2} + \frac{ea}{r^3} - \frac{ear}{(r^2 - a^2)^2}\right\}.$$

7. Find the potential energy due to the mutual action of a charged sphere and the electric mass e at a distance f from its centre.

If M denote the potential energy required,

$$M = \int_f^\infty F dr = \frac{Ee}{f} - \frac{1}{2}\,\frac{e^2 a^3}{f^2(f^2 - a^2)}.$$

The value of M may be obtained otherwise by considering that if U be the potential at A of the distribution on the sphere, by (5), Art. 112, we have

$$U = \frac{E}{f} - \frac{ea^3}{f^2(f^2 - a^2)}.$$

Then, if e be increased by de, the work done against the force exerted by the sphere is Ude; whence $M = \int_0^e Ude$.

From M obtained in this manner the value of F may be deduced. M must not be confounded with the total energy of the electrified system.

8. Show that an uncharged insulated sphere always attracts an electrified particle.

If, in Ex. 6, the charge E be zero,

$$F = \frac{e^2 a^3 (a^2 - 2f^2)}{f^3 (f^2 - a^2)^2},$$

which is always negative.

9. If a sphere be charged and insulated, show that there is one point on each of its radii produced at which an electrified particle is in equilibrium.

In this case, if r be the distance of an electrified particle from the centre, we have

$$\frac{F}{e} = \frac{E}{r^2} - ea^3\,\frac{2r^2 - a^2}{r^3(r^2 - a^2)^2}.$$

Here F is negative when $r = a$. As r increases, F continues to be negative until

$$E - e\,\frac{a}{r}\,\frac{a^2}{r^2 - a^2}\,\frac{2r^2 - a^2}{r^2 - a^2} = 0.$$

There is a point of equilibrium for the value of r satisfying this equation. Since each fraction by which e is multiplied in its left-hand member is diminished

when r is increased, if r increases still more, F becomes, and then always remains, positive. Thus on each radius of the sphere there is one, and only one, point at which an electrified particle is in equilibrium, and at this point the equilibrium is unstable.

10. Find the condition that there should be a circle of zero density on a charged insulated spherical conductor under the influence of an external electrified point A.

By (4), Art. 112, if

$$E + \frac{ea}{f} - \frac{ea(f^2 - a^2)}{(f-a)^3} \quad \text{and} \quad E + \frac{ea}{f} - \frac{ea(f^2 - a^2)}{(f+a)^3}$$

have different algebraical signs, that is, if E lie between

$$e\,\frac{a^2\,(3f-a)}{f\,(f-a)^2} \quad \text{and} \quad -\,e\,\frac{a^2\,(3f+a)}{f(f+a)^2},$$

there must be a circle of zero density the distance r of any one of whose points from A is given by the equation

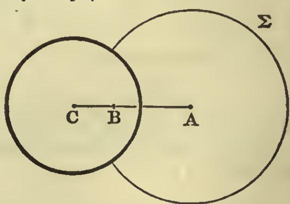
$$E + \frac{ea}{f} - \frac{ea(f^2 - a^2)}{r^3} = 0.$$

11. In Ex. 10, if

$$E = ea\,\left(\frac{1}{\sqrt{(f^2 - a^2)}} - \frac{1}{f}\right)$$

prove that a sphere Σ, described with A as centre and $\sqrt{(f^2 - a^2)}$ as radius, is part of the equipotential surface at which the value of the potential is that on the conductor.

Let $t^2 = f^2 - a^2$; then since $AC\,.\,AB = f\left(f - \frac{a^2}{f}\right) = t^2$, the point B is the image of C in the sphere Σ; also, as



$$E + e\,\frac{a}{f} = \frac{ea}{t},$$

we have

$$e' = -\left(E + \frac{ea}{f}\right)\frac{t}{f},$$

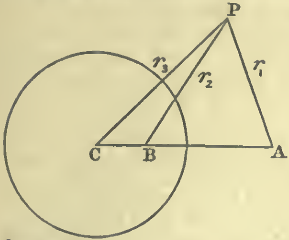
and therefore, by Art. 110, the potential at any point on the sphere Σ is $\frac{e}{t}$, but this is equal to $\left(E + \frac{ea}{f}\right)\frac{1}{a}$, which is the value of the potential at the surface of the conductor.

If L be the value of the potential at the conductor, the equipotential surface for which $V = L$ consists in this case of the surface of the conductor and of the segment external to it of the sphere Σ.

12. In a field of force due to a charged insulated spherical conductor, and

an electrified point A outside it, find the locus of points at which the resultant force is zero.

If B be the image of A in the spherical conductor whose centre is C, and P a point on the required locus, then at P the forces F_1, F_2, and F_3, passing through the points A, B, and C, are in equilibrium, and therefore



$$\frac{F_1}{\sin BPC} = \frac{F_2}{\sin CPA} = \frac{F_3}{\sin APB}.$$

Hence if $AP = r_1$, $BP = r_2$, $CP = r_3$, $CA = f$, the radius of the conductor being a, we have

$$\frac{F_1 r_2 r_3 f}{a^2} = \frac{F_2 r_3 r_1}{f} = \frac{F_3 r_1 r_2 f}{f^2 - a^2},$$

that is

$$\frac{efr_2r_3}{a^2r_1{}^2} = \frac{ear_3r_1}{f^2r_2{}^2} = \frac{(Ef + ea)\,r_1r_2}{(f^2 - a^2)r_3{}^2}.$$

Accordingly the ratios $r_1 : r_2 : r_3$ are given, and therefore the locus of P is the circle in which two spheres intersect, of which one is the surface of the conductor. Accordingly the locus is the circle of zero density, see Ex. 10.

13. A charged insulated spherical conductor is brought into a field of force due to an electrified point A; find the locus of points at which the potential remains unaltered.

Before the conductor is brought into the field the potential at any point P is $\frac{e}{AP}$; afterwards it is given by the equation

$$V = \frac{e}{AP} + \frac{Ef + ea}{f\,.\,CP} - \frac{ea}{f\,.\,BP},$$

where C is the centre of the sphere, and B the image of A; hence the required locus is, in general, a sphere.

14. If the conductor be uncharged, what is the locus in the last example? The plane bisecting CB at right angles.

15. What is the condition that the locus in Example 12 should be a real curve?

16. Determine the distribution of mass on an insulated sphere placed in a field of uniform force.

By (11), Art. 114, the total potential V in external space is given by the equation

$$V = F\left(\frac{a^3}{r^2} - r\right)\cos\theta + K,$$

where θ is the angle which the radius vector r from C makes with the axis of x. Inside the sphere $V = K$. Hence, if σ be the density of the distribution at any point of the surface of the sphere, where $r = a$, we have

$$4\pi\sigma = -\frac{dV}{dr} = 3F\cos\theta, \quad \text{and} \quad \sigma = \frac{3F}{4\pi}\cos\theta.$$

17. An insulated ellipsoïdal conductor is placed in a field of uniform force, one of its axes being in the direction of the force; find the potential in external space of the distribution of electricity induced on the surface of the conductor.

The solution of this question is suggested by the result obtained in Art. 114.

If U denote the potential required, the total mass producing U is zero, $U - Fx =$ constant for all points of the ellipsoidal surface, and U satisfies Laplace's Equation throughout external space, and is zero at infinity. It is plain that X, the expression for the component of the force exerted by a homogeneous ellipsoid whose surface coincides with that of the conductor, satisfies the two latter conditions, and can be made to satisfy the equation $X - Fx = 0$ at the ellipsoidal surface by properly assuming the density ρ. Again, by Ex. 1, Art. 52, the potential X is due to a surface distribution whose density at any point is $l\rho$, where l is the cosine of the angle which the normal makes with the axis of x. Hence the total mass producing this potential is $\rho \int l dS$ taken over the ellipsoid, and this integral is zero as the surface is closed and single sheeted. The function X, therefore, satisfies all the conditions required, and as only one function can do so, $U = X$; and by (12), Art. 87, we have

$$U = 2\pi\rho abcx \int_q^\infty \frac{du}{(a^2+u)^{\frac{3}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}},$$

where ρ is determined by the equation

$$2\pi\rho abc \int_0^\infty \frac{du}{(a^2+u)^{\frac{3}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}} = F.$$

18. An insulated ellipsoidal conductor is charged with a quantity E of electricity, and placed in a field of uniform force, whose direction is parallel to the axis major of the conductor; find the potential U of the distribution on its surface at any point P in external space.

$$\textit{Ans.} \quad U = X + E \int_\lambda^\infty \frac{d\lambda}{\sqrt{(\lambda^2 - h^2)(\lambda^2 - k^2)}},$$

where X is the value of U in Ex. 17, λ the semi-axis major of the ellipsoid passing through P confocal with the conductor, and h and k the constants of the confocal system.

19. An insulated ellipsoidal conductor is placed in a field of uniform force, whose components in the directions of the axes of the ellipsoid are F, F', and F''; find the potential U in external space of the distribution of electricity induced on the surface of the conductor.

If x, y, z denote the coordinates, referred to the axes of the ellipsoid, of a point P in external space, X the component in the direction of x of the force exerted at P by a homogeneous ellipsoid whose surface coincides with that of the conductor, and whose density is ρ, Y' and Z'' the components in the directions of y and z respectively, for two other coincident ellipsoids whose densities are ρ' and ρ'', we have $U = X + Y' + Z''$, where ρ, ρ', and ρ'' are determined by the equations

$$2\pi\rho abc \int_0^\infty \frac{du}{(a^2+u)^{\frac{3}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}} = F,$$

$$2\pi\rho' abc \int_0^\infty \frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{3}{2}}(c^2+u)^{\frac{1}{2}}} = F',$$

$$2\pi\rho'' abc \int_0^\infty \frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{3}{2}}} = F''.$$

20. If the conductor in Ex. 19 be charged with a quantity E of electricity, find the potential U.

$$Ans. \quad U = X + Y' + Z' + E\int_\lambda^\infty \frac{d\lambda}{\sqrt{\{(\lambda^2 - h^2)(\lambda^2 - k^2)\}}},$$

where the notation is the same as that in the preceding Examples.

21. Show that there is only one possible distribution of a given quantity of electric mass on the surface S of an insulated conductor in electric equilibrium under the influence of a given system of electrified points.

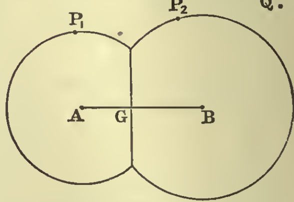
If the distribution on S be in equilibrium its potential U together with v, the potential of the system of electrified points, must be constant on S. Hence $U = C - v$ on S. If there be a second possible distribution having U' for its potential, we have $U' = C' - v$ on S. Then $U - U'$ is constant on S, and is the potential of a distribution of mass whose total amount is zero. Hence, by Art. 62, the potential $U - U'$ is zero for the whole of space, and consequently, by Art. 46, the corresponding surface density is everywhere zero. Accordingly, the two supposed distributions on S are identical.

115. **Spheres Cutting Orthogonally**.—If a field of force be due merely to a charged insulated spherical conductor S_1, the potential is everywhere the same as if the charge on this conductor were concentrated at its centre A. If another charged spherical conductor S_2 be brought into the field, the distribution on S_2 could be found by the method of Arts. 111, 112, if the electric mass on S_1 were rigidly fixed. As this is not the case, applications of the method of Art. 111, would, in general, have to be repeated *ad infinitum*. If, however, the image of A in S_2 coincide with the image in S_1 of B, the centre of S_2, the distributions on the two spherical surfaces can be readily obtained.

In this case, if the distance AB be denoted by c, the radii of the spheres by a and b, and the distances of the double image G from A and B by ξ and η, we have $c\xi = a^2$, $c\eta = b^2$, $\xi + \eta = c$; whence eliminating ξ and η, we get $c^2 = a^2 + b^2$, that is, the spheres cut orthogonally.

Since the spheres intersect in real points, they must form a continuous conductor, and we are led to the consideration of the problem, to find the distribution and total charge on an insulated conductor which is formed of the larger segments of two spheres cutting orthogonally, and which is at a given potential L.

A potential L on the surface of the sphere S_1 is produced by a charge La at A, and a charge Lb at B produces a potential L at S_2, also a charge $-\dfrac{Lab}{c}$ at G neutralizes the effect of La at S_2, and that of Lb at S_1. Hence the charges which have been enumerated produce a potential L all over the surface of the conductor, and therefore, by Art. 64, produce the same potential in external space as the actual distribution.



We conclude, therefore, that inside the surface of the conductor the value of the potential is everywhere L, and that at any point Q in external space, the potential V is given by the equation

$$V = \frac{La}{AQ} + \frac{Lb}{BQ} - \frac{Lab}{c.GQ}. \qquad (12)$$

The densities σ_1 and σ_2 of the distribution at points P_1 and P_2 on the two spherical surfaces, and the total mass E on the conductor, are given by the equations

$$\sigma_1 = \frac{L}{4\pi a}\left(1 - \frac{b^3}{BP_1{}^3}\right), \quad \sigma_2 = \frac{L}{4\pi b}\left(1 - \frac{a^3}{AP_2{}^3}\right), \qquad (13)$$

$$E = L\left(a + b - \frac{ab}{c}\right). \qquad (14)$$

116. **Total Mass on Conductor.**—When the potential on one side of the surface of a conductor in equilibrium is the same as that due to a set of electrified points, the total mass on any portion S of this surface of the conductor can be easily deduced from the total induction over S.

Let e_1, e_2, &c. be the charges at the points A_1, A_2, &c. which can produce the actual potential at one side of S; N the resultant force in the direction of the normal drawn towards this side at any point of S whose distances from A_1, &c. are r_1, &c.; ψ_1, &c. the angles which r_1, &c. make with this normal; Ω_1, &c. the solid angles which S or its bounding

curve subtends at A_1, &c.; and E the total charge required; then

$$4\pi E = 4\pi \int \sigma dS = \int N dS = \int \left(\frac{e_1}{r_1^2} \cos \psi_1 + \frac{e_2}{r_2^2} \cos \psi_2 + \&c. \right) dS$$

$$= e_1 \Omega_1 + e_2 \Omega_2 + \&c.; \text{ whence}$$

$$E = \frac{e_1 \Omega_1 + e_2 \Omega_2 + \&c.}{4\pi}. \tag{15}$$

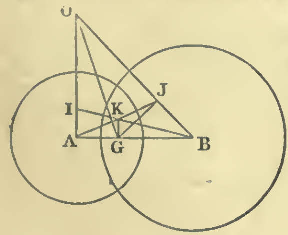
If S be a closed surface, Ω is equal to 4π or zero, according as A is inside or outside the surface, and the total mass on S is equal to the sum of the charges at the internal points.

Examples.

1. An insulated conductor formed of the larger segments of two spheres cutting orthogonally is in electric equilibrium: what is the density of the distribution at a point on the circle of intersection of the spheres? *Ans.* 0.

2. A conductor formed of the larger segments of two spheres cutting orthogonally, whose centres are A and B, is at potential zero under the influence of an external electrified point O: find the potential at any point, and the distribution of mass on the conductor.

Let I and J be the images of O in the spheres, then AJ and BI intersect at a point K such that



$$AK \cdot AJ = a^2, \quad BK \cdot BI = b^2,$$

where a and b are the radii of the spheres. For, if G be the point in which AB meets the plane of intersection of the spheres, the quadrilaterals $BGIO$ and $AGJO$ are cyclic; whence angle $GBI = GOI = AJG$, and $GKJB$ is cyclic, and $AK \cdot AJ = AG \cdot AB = a^2$. In like manner $BK \cdot BI = b^2$.

Hence charges $-\dfrac{ea}{AO}$ at I, $-\dfrac{eb}{BO}$ at J,

and $$\frac{eab}{AJ \cdot BO} \text{ or } \frac{eab}{BI \cdot AO} \text{ at } K$$

produce a potential which is zero at the surface of the conductor, the charge at O being e. It is easy to see that $AJ \cdot BO = BI \cdot AO$, since the quadrilateral $IOJK$ is cyclic and therefore the angle $AJO = BIA$.

If we put $AO = f_1$, $BO = f_2$, and express $AJ \cdot BO$ in terms of f_1, f_2, a, and b, we find $AJ^2 \cdot BO^2 = a^2 f_2^2 + b^2 f_1^2 - a^2 b^2$.

Hence, as the values of the charges at I, J, and K, we obtain

$$-\frac{ea}{f_1}, \quad -\frac{eb}{f_2}, \quad \text{and} \quad \frac{eab}{\sqrt{(a^2f_2^2 + b^2f_1^2 - a^2b^2)}};$$

and if V denote the potential at any point Q in external space, we have

$$V = e\left\{\frac{1}{OQ} - \frac{a}{f_1}\frac{1}{IQ} - \frac{b}{f_2}\frac{1}{JQ} + \frac{ab}{\sqrt{(a^2f_2^2 + b^2f_1^2 - a^2b^2)}}\frac{1}{KQ}\right\}.$$

The density σ of the distribution at any point P on the sphere whose centre is A is given by the equation

$$4\pi a\sigma = e\,(a^2 - f_1^2)\left\{\frac{1}{OP^3} - \frac{b^3}{f_2^3}\frac{1}{JP^3}\right\}.$$

At any point on the circle of intersection of the two spheres, σ is zero.

3. A hollow conductor formed of the smaller segments of two spheres cutting orthogonally is at potential zero under the influence of an internal electrified point O; find the potential at any point in the interior region and the distribution of mass on its bounding surface.

Adopting the notation of the last Example, we have at any point Q in the interior region the same expression for V as that given in the last Example. For the density σ of the distribution at any point P of the interior surface of the sphere whose centre is A, we get

$$4\pi a\sigma = -\,e\,(a^2 - f_1^2)\left\{\frac{1}{OP^3} - \frac{b^3}{f_2^3}\frac{1}{JP^3}\right\}.$$

4. In Ex. 2 if the conductor be at potential L, find the distribution of mass on its surface, and the potential V at any point Q in external space,

$$V = \frac{La}{AQ} + \frac{Lb}{BQ} - \frac{Lab}{cGQ} + \frac{e}{OQ} - \frac{a}{f_1}\frac{e}{IQ} - \frac{b}{f_2}\frac{e}{JQ} + \frac{ab}{\sqrt{(a^2f_2^2 + b^2f_1^2 - a^2b^2)}}\frac{e}{KQ}.$$

The density σ of the distribution at any point P on the sphere, whose centre is A is given by the equation

$$4\pi a\sigma = L\left(1 - \frac{b^3}{BP^3}\right) + e\,(a^2 - f_1^2)\left\{\frac{1}{OP^3} - \frac{b^3}{f_2^3}\frac{1}{JP^3}\right\}.$$

5. In the last Example, if the total charge E on the conductor be given, determine the potential and the distribution of mass.

Here L is found from the equation

$$E = L\left(a + b - \frac{ab}{c}\right) + \frac{eab}{\sqrt{(a^2f_2^2 + b^2f_1^2 - a^2b^2)}} - \frac{ea}{f_1} - \frac{eb}{f_2}.$$

Hence we obtain V and σ as in the last Example.

6. The larger segments of two spheres which cut orthogonally are formed of conducting material, and held together in perfect contact along their common circle by a rigid rod joining the centres of the spheres. The conductor so formed is insulated and charged to potential L; find the stress on the connecting rod. (See figure, Art. 115.)

The mutual force between the two spherical segments is the same as that which the whole conductor exerts on one segment. Adopting the notation of Art. 115, let ϕ be the angle which AP_1 makes with AB, and let $BP_1 = r$, then $r^2 = a^2 + c^2 - 2ac\cos\phi$; whence $rdr = ac\sin\phi d\phi$, and if dS be the element of surface between two consecutive small circles of the sphere having A for centre, whose poles lie on AB, we have

$$dS = 2\pi a^2 \sin\phi d\phi = \frac{2\pi a}{c} rdr.$$

Also by (8), Art. 35, and (13), Art. 115, the force which the conductor exerts on the element dS is

$$\frac{L^2}{8\pi a^2}\left(1 - \frac{b^3}{r^3}\right)^2 dS.$$

Resolving this along AB, and substituting for $-\cos\phi$ its value $\dfrac{r^2 - a^2 - c^2}{2ac}$, we get for X the required stress

$$\frac{L^2}{8a^2c^2}\int_b^{a+c}\left(1 - \frac{b^3}{r^3}\right)^2 (r^2 - a^2 - c^2)\, rdr.$$

If we perform the integration we obtain for $\dfrac{8a^2b^2c^2X}{L^2}$, an expression in a, b, and c, which, by means of the equation $c^2 = a^2 + b^2$, can be reduced to the form

$$2(a^3 + b^3 - c^3)^2. \quad \text{Hence} \quad X = L^2\frac{(a^3 + b^3 - c^3)^2}{4a^2b^2c^2}.$$

7. A charged insulated uninfluenced conductor formed of the larger segments of two spheres cutting orthogonally is at potential L; find the total charge on the spherical surface whose radius is a.

The solid angle subtended at A (see fig. of Art. 115) by the spherical surface is $2\pi\left(1 + \dfrac{GA}{a}\right)$, and those subtended at B and G are $2\pi\left(1 - \dfrac{GB}{b}\right)$ and 2π; hence, if E_a be the total charge, we have

$$E_a = \frac{La}{2}\left(1 + \frac{GA}{a}\right) + \frac{Lb}{2}\left(1 - \frac{GB}{b}\right) - \frac{Lab}{2c}$$

$$= \frac{L}{2}\left\{a + b - \frac{ab}{c} + (a - b)\frac{a + b}{c}\right\}.$$

8. A large insulated spherical conductor with a small hemispherical boss on its surface is charged to potential L: find the mean density of the distribution on the hemisphere, and compare it with that on the sphere.

If E_a and E_b be the total charges on the boss and on the sphere, their radii being a and b, we have

$$E_a = \tfrac{1}{2} L \left\{ a + b - a \left(1 + \frac{a^2}{b^2}\right)^{-\frac{1}{2}} - b\left(1 - \frac{a^2}{b^2}\right)\left(1 + \frac{a^2}{b^2}\right)^{-\frac{1}{2}} \right\}$$

$$= \frac{Lb}{2}\left\{\frac{a^2}{b^2} + \frac{1}{2}\frac{a^2}{b^2}\right\} = \frac{3}{4}\frac{La^2}{b},$$

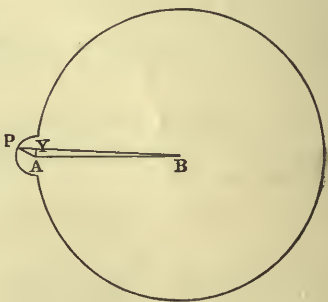
the higher powers of $\frac{a}{b}$ being neglected. The mean density σ_a of the distribution on the boss is given then by the equation

$$\sigma_a = \frac{E_a}{2\pi a^2} = \frac{3}{8\pi}\frac{L}{b}.$$

Again, $E_b = Lb$, approximately, and

$$\sigma_b = \frac{E_b}{4\pi b^2} = \frac{L}{4\pi b}. \quad \text{Hence} \quad \sigma_a = \frac{3}{2}\sigma_b.$$

9. Find the density of the distribution at any point on either surface in the last Example.



If σ_1 be the density at any point P on the hemisphere, and σ_2 that at any point Q of the sphere, by (13), Art. 115,

$$\sigma_1 = \frac{L}{4\pi a}\left(1 - \frac{b^3}{BP^3}\right).$$

Let fall a perpendicular AY on BP, then $BP = BY + a\cos\theta$, where θ is the angle BP makes with the normal to the hemisphere at P, and if $\left(\frac{a}{b}\right)^2$ be neglected $BY = BA = b$; whence

$$\frac{b^3}{BP^3} = 1 - 3\frac{a}{b}\cos\theta,$$

and

$$\sigma_1 = \frac{3L}{4\pi b}\cos\theta = 3\sigma_b\cos\theta.$$

Again,

$$\sigma_2 = \frac{L}{4\pi b}\left(1 - \frac{a^3}{AQ^3}\right).$$

Except in the vicinity of A, the value of $\frac{a^3}{AQ^3}$ is insensible, and $\sigma_2 = \sigma_b$.

10. A conductor composed of an infinite plane and a hemisphere whose centre A is on the plane, is charged with electricity: find the ratio of the mean densities of the distributions on the hemisphere and plane. *Ans.* $\frac{3}{2}$.

11. In the last Example, find the density of the distribution at any point on either surface.

If F be the resultant forc:, and σ_0 the density, at a point on the plane at a long distance from A, σ_1 the density at a point on the hemisphere where the normal makes an angle θ with the perpendicular on the plane, and σ_2 the density at a point Q on the latter in the vicinity of A, we have

$$\sigma_0 = \frac{F}{4\pi}, \quad \sigma_1 = 3\sigma_0 \cos\theta, \quad \sigma_2 = \sigma_0 \left(1 - \frac{a^3}{AQ^3}\right).$$

12. The equation of a closed surface S can be expressed in the form

$$\frac{l}{r_1} + \frac{m}{r_2} + \frac{n}{r_3} = \frac{k}{c},$$

where r_1, r_2, r_3 denote the distances of any point from three fixed points A, B, C inside S, and where l, m, n, k, and c denote constant magnitudes. If a conductor, whose surface is S, be insulated and charged to potential L, find its potential V at any point in external space.

Ans. $V = \dfrac{Lc}{k}\left(\dfrac{l}{r_1} + \quad_{r_2} + \dfrac{n}{r_3}\right).$

117. Uniplanar Distribution. Image of Point in Circle.—For uniplanar distributions of mass, acting with a force varying inversely as the distance, the theory of images differs in some respects from that belonging to distributions in space of three dimensions.

If A and B be inverse points with respect to a circle whose centre is C, and P any point on its circumference s, we have, as in Art. 110, $AP : BP :: CA : CP$; whence $AP : BP$ is constant for all points on s, and therefore so also is $\log \dfrac{AP}{BP}$.

Hence, if there be a mass e at A and a mass $-e$ at B, the potential due to these two conjointly is constant for all points of s; and $-e$ is the total mass corresponding to a distribution on s producing in external space the same potential as $-e$ at B.

We can now see that if there be an insulated circle s, whose centre is C, and on which there is a charge E in presence of an external point A at which the mass e is concentrated, the potential in the region outside s, due to the distribution on it, is the same as that due to a charge $-e$ at B, the point inverse to A with respect to s, together with a charge $E + e$ at C. For this is the only possible potential

due to a distribution of mass E on s which, along with that due to e at A, produces a total potential constant at s.

This result is proved in a manner similar to that employed in Art. 112. If, as in that Article, U and U' be two different possible potentials of the distributions on s, and $\phi = U - U'$, it is not in the present case immediately obvious that $\int \phi \frac{d\phi}{d\nu}\, ds$ taken round the circle at infinity is zero. This, however, follows from the consideration that U and U' are each due to the same total mass E.

It appears from the preceding investigation that corresponding charges at points which are the images of each other in a circle produce, at its circumference, a potential which is constant, but not zero.

In considering this apparent anomaly, it is to be remembered that a uniplanar distribution is not a physical reality, but a mathematical artifice to simplify problems having to do with a cylindrical distribution.

If V denote the three-dimensional potential at a point P of a straight thin bar of uniform linear density λ, of length b, and terminated by the perpendicular p on it from P, and if r be the distance of P from the other extremity of this line, and θ the angle which r makes with p, it is easy to see that

$$V = \lambda \log \frac{1 + \sin\theta}{\cos\theta} = \lambda \log \frac{r + b}{p}.$$

If we now suppose the length b to become infinite, we have

$$V = \lambda \log \frac{2b}{p}.$$

Accordingly, if V be the potential at any point P due to a thin bar whose total length is l and which is infinite in both directions, we have $V = 2\lambda \log \frac{l}{p}$. If dS be the section of the thin bar, and ρ the volume density, $\lambda = \rho dS$. Hence, as by Art. 11, $2\rho = \tau$, the potential at P of a cylindrical distribution, corresponding to the uniplanar mass e concentrated at a point A, is $e \log_e \frac{l}{AP}$, which is infinite when e is finite.

The potential of a cylindrical distribution corresponding to two equal uniplanar charges e, one positive, the other negative, at points A and B, is $e \log \frac{BP}{AP}$, which is the same as the uniplanar potential and is finite.

Hence, if P be a point on a circle with respect to which A and B are images, an infinitely small uniplanar charge, at the centre of the circle or uniformly distributed on its circumference, corresponds to a uniform distribution on the surface of the infinite cylinder, of which the circle is a section, sufficient to reduce the total three dimensional potential at this surface to zero.

Thus it is seen that the theory of uniplanar images is in accordance with the general theory of electrical distributions.

If we now denote by V the total uniplanar potential due to e at A and to the charge E on the circle, we have at any point Q in external space

$$V = e \log \frac{BQ}{AQ} + (E + e) \log \frac{1}{CQ}. \qquad (16)$$

Throughout the region inside the circle V is constant, and has the value

$$e \log \frac{a}{f} + (E + e) \log \frac{1}{a},$$

where a denotes the radius of the circle and f the distance of A from the centre.

If the point A at which the influencing charge e is situated be inside the circle, the potential V is constant at the circumference of the circle, its value at infinity is

$$(E + e) \log \frac{1}{CQ},$$

and the value of the integral

$$\int \frac{dV}{d\nu} ds$$

taken round the circle is $-2\pi(E+e)$. Also at all points outside the circle, $\nabla^2 V = 0$. Hence, throughout this region

$$V = (E+e)\log\frac{1}{CQ}, \tag{17}$$

as there can be only one function satisfying the conditions specified above.

Inside the circle the potential U of the distribution on its circumference must satisfy the conditions $\nabla^2 U = 0$ throughout the interior region, and

$$U + e\log\frac{1}{AP} = \text{constant}$$

for any point P on the circumference.

Any two functions of the coordinates satisfying these conditions can differ only by a constant. Hence

$$U = e\log BQ + c.$$

The constant c is determined from the value of V at the circumference. From this we have

$$e\log\frac{BP}{AP} + c = (E+e)\log\frac{1}{a};$$

whence

$$c = (E+e)\log\frac{1}{a} - e\log\frac{a}{f},$$

and the total potential V at any point Q in the interior of the circle is given by the equation

$$V = e\left(\log\frac{BQ}{AQ} - \log\frac{a}{f}\right) + (E+e)\log\frac{1}{a}. \tag{18}$$

The formulæ for a cylindrical distribution of mass in which e and E denote charges per unit of length are obtained from those for the corresponding uniplanar distribution by changing e and E into $2e$ and $2E$.

118. **Total Uniplanar Mass on Curve.**—If the potential on one side of a curve be constant, and on the other side be the same as that due to a number of charged points

A_1, A_2, &c., the total mass on any portion s of this curve can be found in a manner similar to that employed in Art. 116. If E be the total mass on s, we have in this case

$$2\pi E = 2\pi \int v ds = \int N ds = \int \left(\frac{e_1}{r_1} \cos \psi_1 + \frac{e_2}{r_2} \cos \psi_2 + \text{\&c.} \right) ds$$

$$= e_1\theta_1 + e_2\theta_2 + \text{\&c.},$$

where θ_1, &c., are the angles which s subtends at A_1, &c.; whence

$$E = \frac{e_1\theta_1 + e_2\theta_2 + \text{\&c.}}{2\pi}. \tag{19}$$

Examples.

1. An insulated circle having a charge E is influenced by a quantity e of uniplanar mass concentrated at a point A; find the distribution of mass on the circumference of the circle.

At a point on the circle whose distance from A is r, the density v of the line distribution is given by the equation

$$2\pi a v = E + e + e\,\frac{a^2 - f^2}{r^2},$$

if the point A be outside the circle.

If A be inside, the equation for v becomes

$$2\pi a v = E + e - e\,\frac{a^2 - f^2}{r^2}.$$

2. Show that the distribution on the circle is the same whether the influencing mass e be situated at the point A or at its image B.

Let f' and r' be the distances of B from the centre and from any point on the circumference, then

$$f^2 f'^2 = a^4, \quad \text{and} \quad \frac{r^2}{r'^2} = \frac{f^2}{a^2};$$

whence

$$\frac{a^2 - f^2}{r^2} = -\,\frac{a^2 - f'^2}{r'^2},$$

and the required result follows from Ex. 1.

3. In Ex. 1 find the potential U of the mass distributed on the circumference of the circle.

Here let A be the exterior of the two inverse points at one of which e is situated; then at any point Q outside the circle $U = e \log BQ - (E + e) \log CQ$,

and at any internal point $U = e(\log AQ - \log f) - E \log a$, where f is the distance of A from the centre C.

4. In Ex. 1, if the charge on the circle be $-e$; find the potential V at its circumference.

If A be outside the circle, $V = e \log \frac{a}{f}$; if A be inside, $V = 0$. This case of a uniplanar distribution of mass corresponds to that of a sphere put to earth under the influence of an electrified point. For the sphere, whether A be an internal or an external point, the potential at the surface is zero, but the total mass on the surface is different in the two cases. For the circle, the total mass on the circumference is the same for either position of A, but the potential is different.

5. If uniplanar mass be distributed on the circumference of a circle, so that the density at any point varies inversely as the square of its distance from a fixed point, show that the distribution is centrobaric, and find the baric centre.

6. Find the value of the integral $\int \frac{f^2 - a^2}{r^2} ds$, taken round the circumference s of a circle whose radius is a, where r and f are the distances of ds, and the centre of the circle from an external point A. *Ans.* $2\pi a$.

7. If A be inside the circle, what is the value of the integral in the last Example? *Ans.* $-2\pi a$.

8. Find the force which an insulated circle, having a uniplanar charge E, exerts on a quantity e of uniplanar mass concentrated at an external point A.

If F denote the force, and r the distance of A from the centre of the circle

$$F = e\left\{\frac{E+e}{r} - \frac{er}{r^2 - a^2}\right\}.$$

9. Uniplanar mass is distributed on the boundary of the larger segments of two circles cutting orthogonally so as to produce a uniform potential L; find the potential at any point in external space, and the distribution of the mass (see fig., Art. 115).

Let A and B be the centres of the two circles, and G the point of intersection of their common chord with AB. If we suppose equal quantities η of uniplanar mass placed at A and B, and a quantity $-\eta$ placed at G, the potential at the boundary of one of the circles is $\eta \log \frac{1}{c}$, where c is the distance between their centres. Hence, if η be determined by the equation $\eta \log \frac{1}{c} = L$, the potential V at any point Q in external space is given by the equation

$$V = \eta(\log GQ - \log AQ - \log BQ),$$

and v the density of the distribution at any point P of the circle whose centre is A by the equation

$$2\pi a v = \eta \left(1 - \frac{b^2}{BP^2}\right) = \frac{L}{\log \frac{1}{c}} \left(1 - \frac{b^2}{BP^2}\right).$$

At a point of intersection of the circles $v = 0$.

The total mass on the boundary is η.

10. In the last example, if the potential L be due partly to the distribution on the circular boundary and partly to uniplanar mass e concentrated at an external point O, find the potential in external space, and the distribution of mass.

Adopting the notation of Ex. 2, Art. 116, if η be the hypothetical charge at A or B, we have

$$L = e \left(\log \frac{a}{f_1} - \log \frac{a}{AJ}\right) + \eta \log \frac{1}{c}$$

$$= e \left\{\tfrac{1}{2} \log (a^2 f_2^2 + b^2 f_1^2 - a^2 b^2) - \log f_1 - \log f_2\right\} + \eta \log \frac{1}{c},$$

which determines η. The potential V at an external point Q is given, then, by the equation

$$V = e\,(\log IQ + \log JQ - \log OQ - \log KQ) + \eta\,(\log GQ - \log AQ - \log BQ).$$

If v be the density of the distribution at any point P of the circle whose centre is A, we have

$$2\pi a v = \eta \left(1 - \frac{b^2}{BP^2}\right) + e\,(a^2 - f_1^2) \left\{\frac{1}{OP^2} - \frac{b^2}{f_2^2} \frac{1}{JP^2}\right\}.$$

11. In Example 9 find the total mass on the arc of the circle whose centre is A.

Let α and β be the angles which the common chord subtends at the centres A and B of the circles, then if E_a be the total mass required, we have

$$E_a = \eta\, \frac{2\pi - \alpha + \beta - \pi}{2\pi} = \frac{\eta}{2} \left(1 - \frac{\alpha - \beta}{\pi}\right).$$

119. **Inversion.**—The theory of images has suggested a transformation by means of which problems as to distributions of electricity can in many cases be much simplified. This process is called *Electrical Inversion*, and may be described as follows.

If, with any point O as centre, a sphere of radius R be described, the images with respect to this sphere of a system of electrified points form a new system related to the former,

so that if the one be assigned the other can be determined. In the case of images physically related, if one charge be positive the other is negative. Here, however, the relation of the two systems is purely mathematical, and we may therefore attribute the same algebraical sign to the original charge and to its image. The two electrified systems are then called *inverse* systems with respect to the origin O, and the arbitrary length R is called *the radius of inversion.*

If A be a point of one of the systems at which there is a charge e, and A' and e' be the corresponding point and charge of the inverse system, A' is on the line OA, and $OA \cdot OA' = R^2$, also

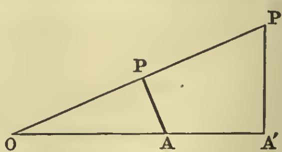
$$e' = \frac{R}{OA}\, e = \frac{OA'}{R}\, e.$$

If we denote the systems by E and E', it is easy to prove the following propositions:—

1°. Corresponding points in the two systems lie on curves or surfaces which are inverse with respect to O. Hence a sphere S in the system E corresponds to a sphere S' in the system E', except O be on the surface S, in which case the corresponding surface in E' is a plane.

In general a plane corresponds to a sphere passing through O, but if O be on the plane, then the corresponding surface is the same plane.

2°. If V_P be the potential of the system E at any point P, and V'_P the potential of E' at the point P' corresponding to P, then,



$$V'_{P'} = \frac{RV_P}{OP'}. \qquad (20)$$

For,

$$V'_{P'} = \Sigma \frac{e'}{P'A'} = \Sigma \frac{Re}{OA \cdot P'A'},$$

but from the similar triangles AOP and $P'OA'$, we have $OA \cdot P'A' = OP' \cdot PA$; whence

$$V' = \frac{R}{OP'} \Sigma \frac{e}{PA} = \frac{R}{OP'} V_P.$$

(*a*) Hence, if the potential of E at P be zero, so also is the potential of E' at P'.

(*b*) If the potential of E have a constant value L at all points of a curve or surface, the potential of E' at any point P' on the inverse curve or surface is that due to the mass RL placed at the origin. Hence the potential at P' due to E' together with a mass $-RL$ at O is zero.

Thus, by means of inversion, a distribution producing a constant potential can be transformed into a distribution producing a potential zero.

3°. If ρ be the density of a volume distribution at any point A of the system E, and ρ' the corresponding density in the inverse system,

$$\rho' = \left(\frac{r}{R}\right)^5 \rho = \left(\frac{R}{r'}\right)^5 \rho \tag{21}$$

where $r = OA$, and $r' = OA'$.

To prove this, we have

$$\rho' d\mathfrak{S}' = e' = \frac{R}{r} e = \frac{R}{r} \rho d\mathfrak{S};$$

but

$$d\mathfrak{S}' = r'^2 dr' d\omega = \frac{R^4}{r^2} \frac{R^2}{r^2} dr\, d\omega,$$

and $d\mathfrak{S} = r^2\, dr\, d\omega$; substituting and reducing, we have the required result.

4°. If σ be the density of a surface distribution at any point A of the system E, and σ' the density at the corresponding point A' of the inverse system,

$$\sigma' = \left(\frac{r}{R}\right)^3 \sigma = \left(\frac{R}{r'}\right)^3 \sigma. \tag{22}$$

This is proved in a manner similar to that employed for volume densities by means of the equations

$$dS \cos \psi = r^2\, d\omega, \qquad dS' \cos \psi = r'^2\, d\omega,$$

where ψ is the angle which the radius vector makes with the normal to either of the inverse surface elements dS and dS'.

5°. If two points A_1 and A_2 belonging to the system E are images of each other in a surface S, the inverse points A'_1 and A'_2 are images of each other in the surface S' which is the inverse of S.

This follows from the consideration that the joint potential due to e_1 at A_1 and e_2 at A_2 is zero at S, and therefore, by 1°, (a) so also is the potential at S' due to e'_1 at A'_1 and e'_2 at A'_2. Hence, A'_1 and A'_2 are images of each other with respect to S'.

As a particular case of the above, we have the result that, if A be the centre of the sphere S, the point A' is the image of the origin in the surface S' which is the inverse of S. This is obvious if we remember that the image of A in S is a point at infinity whose inverse is the origin.

6°. If t and t' be the tangents from O to inverse spheres whose radii are a and a', and the distances of whose centres from O are α and α', we have

$$tt' = R^2, \quad \frac{a}{a'} = \frac{\alpha}{\alpha'} = \frac{t}{t'} = \frac{R^2}{t'^2} = \frac{R^2}{\alpha'^2 - a'^2} = \frac{t^2}{R^2} = \frac{\alpha^2 - a^2}{R^2}. \qquad (23)$$

7°. If p be the perpendicular distance of O from a plane whose inverse is a sphere having a for radius, a is given by the equation

$$2pa = R^2. \qquad (24)$$

EXAMPLES.

1. Find the distribution of mass on a sphere whose centre is A, and which is at potential zero under the influence of a charge e at a point O.

Invert the sphere from O, and we obtain a sphere at constant potential L, where $RL = -e$. The density σ' at any point P' of this sphere is given by the equation

$$\sigma' = \frac{L}{4\pi a'} = -\frac{e}{4\pi a' R}, \quad \text{then, by (22),} \quad \sigma = -\frac{R^2}{OP^3}\frac{e}{4\pi a'},$$

$$\text{but, by (23),} \qquad \frac{a}{a'} = \frac{OA^2 - a^2}{R^2}, \qquad \text{whence} \qquad \sigma = -\frac{e}{4\pi a}\frac{OA^2 - a^2}{OP^3}.$$

2. Show that a solid sphere S whose density varies inversely as the fifth power of the distance r from a point O is centrobaric, and find its baric centre.

Invert from O; then, in the inverse sphere S'

$$\rho' = \left(\frac{r}{R}\right)^5 \rho = \left(\frac{r}{R}\right)^5 \frac{K}{r^5} = \frac{K}{R^5};$$

therefore the potential of S' has a constant value L at its surface; and a mass RL at O produces at the surface of S a potential equal to that of S itself. Hence O is the baric centre if it be inside the surface of S. If it be outside, the baric centre is the image of O in this surface.

3. Two planes, cutting at right angles and terminated by their line of intersection, are at potential zero under the influence of a charge e situated at a point O between them; find the distribution of mass on the planes, and the potential at any point.

If we draw perpendiculars p_1 and p_2 from O on the planes, take, at the other side of the planes on these perpendiculars, points I and J at the same distances from the planes as O, and draw through I and J, in the plane of OI and OJ, parallels, meeting in K, to the planes; charges, whose amounts are each $-e$, placed at I and J, together with e at K, produce along with e at O a potential zero on each of the planes. Hence, at any point Q in the region containing O, the potential V is given by the equation

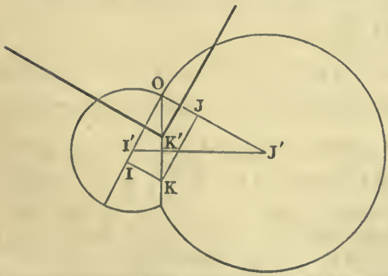
$$V = e\left(\frac{1}{OQ} + \frac{1}{KQ} - \frac{1}{IQ} - \frac{1}{JQ}\right),$$

and in the region separated by the planes from O the potential is zero.

At any point P in the plane whose distance from O is p_1, the density σ_1 is given by the equation

$$\sigma_1 = -\frac{ep_1}{2\pi}\left(\frac{1}{OP^3} - \frac{1}{JP^3}\right).$$

4. Find, by inversion, the distribution of mass on the larger segments of two spheres cutting orthogonally and at constant potential. (For the direct solution of this problem see Art. 115.)



If we invert from O we obtain as the inverse of the planes in Ex. 3, the larger segments of two spheres cutting orthogonally, and at constant potential L, where $RL = -e$, also, if a be the radius of the sphere which is the inverse of the plane the perpendicular on which is p_1, we have $2ap_1 = R^2$, and if σ'_1 be the density of the distribution of mass at any point P' of this sphere,

$$\sigma'_1 = \left(\frac{OP}{R}\right)^3 \sigma_1 = \frac{L}{4\pi a}\left(1 - \frac{OP^3}{JP^3}\right),$$

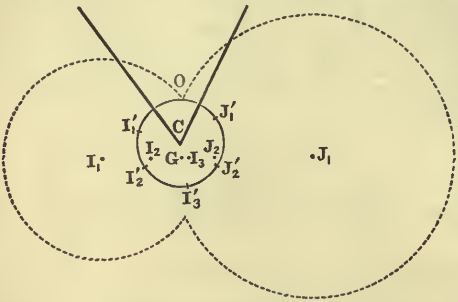
but $\dfrac{OP}{JP} = \dfrac{OJ'}{J'P'}$, and J' is the centre of the sphere inverse to the plane whose perpendicular is p_2. Hence if b denote the radius of this sphere, $OJ' = b$, and

$$\sigma'_1 = \frac{L}{4\pi a}\left(1 - \frac{b^3}{J'P'^3}\right).$$

5. A conductor, formed of the larger segments of two spheres which intersect at an angle of 60°, is insulated and charged to potential L; find the distribution of mass on the conductor, and the potential at any point.

Let the spheres be denoted by (A) and (B), take any point O on their circle of intersection, and invert the system from this point. We have, then, two planes (A') and (B') intersecting at an angle of 60°, and at potential zero under the influence, at O, of a charge $-RL$, which may be denoted by e'. The plane passing through O and the centres of the spheres, which we may call (O), meets the planes (A') and (B') perpendicularly, and contains a series of images charges at which, along with e' at O, produce the actual potential in the region between the planes and containing O. Hence, by 2°, Art. 119, the inverse system of images produce a potential in external space which is the same as that due to the actual distribution on the conductor.

Let C be the point in which (A') and (B') intersect (O); then the images of O in (A') and (B') lie on a circle in the plane (O), having C as centre and CO as radius. Let I'_1 be the image of O in (A'), J'_1 its image in (B'), I'_2 the image of J'_1 in (A'), J'_2 the image of I'_1 in (B'), and so on, and let the angular distances of I'_1, &c., from CO be denoted by i'_1, &c., then, if CO make angles α and β with (A') and (B'), we have $i'_1 = 2\alpha$, $j'_1 = -2\beta$, $i'_2 = \alpha - j'_1 + \alpha = 2(\alpha+\beta)$,



$$j'_2 = -\beta - (i'_1 + \beta) = -2(\alpha+\beta), \quad i'_3 = 2\alpha - j'_2 = 2\alpha + 2(\alpha+\beta),$$

$$j'_3 = -2\beta - i'_2 = -2\beta - 2(\alpha+\beta).$$

Now $\alpha + \beta = \frac{\pi}{3}$; hence, $i'_3 - j'_3 = 2\pi$, and the points I'_3 and J'_3 coincide. The charges are each $-e'$ at images having odd suffixes, and $+e'$ at those having even ones.

In the inverse system, the points inverse to I'_1 and J'_1 are the centres of the spheres (A) and (B). The charge at the centre I_1 is $-e'\,\frac{OI_1}{R}$, or aL, where a is the radius of the sphere (A). Since the images I'_1, J'_1, &c., lie on a circle through O, the inverse points lie on the straight line joining the centres I_1 and J_1, and are successive images of them in the spheres (B) and (A).

Also, if the charges at the points I_1, &c., be denoted by the same letters, and the radius of the sphere (B) by b, we have

$$I_1 = aL, \quad J_1 = bL, \quad \frac{I_1}{OI_1} + \frac{J_2}{OJ_2} = 0, \quad \frac{J_1}{OJ_1} + \frac{I_2}{OI_2} = 0, \text{ \&c.};$$

whence,

$$\frac{I_1}{OI_1} = \frac{-J_2}{OJ_2} = \frac{I_3}{OI_3} = \frac{-I_2}{OI_2} = \frac{J_1}{OJ_1}.$$

The distances I_1I_2, &c., are given by the equations

$$I_1I_2 = \frac{a^2}{c}, \quad J_1J_2 = \frac{b^2}{c}, \quad I_1I_3 = \frac{a^2c}{c^2 - b^2}, \quad J_1J_3 = \frac{b^2c}{c^2 - a^2} \text{ where } c = I_1J_1;$$

the surface density σ at any point P on (A) by the equation

$$4\pi\sigma = \frac{I_1}{a^2} + \frac{J_1(a^2 - c^2)}{aJ_1P^3} + \frac{J_2(a^2 - I_1J_2^2)}{aJ_2P^3};$$

and the potential V at any point Q in external space by the equation

$$V = \frac{I_1}{I_1Q} + \frac{J_1}{J_1Q} + \frac{I_2}{I_2Q} + \frac{J_2}{J_2Q} + \frac{I_3}{I_3Q}.$$

If E_a be the total charge on the sphere whose radius is a, by (15), Art. 116, as in Ex. 7, Art. 116, we have

$$E_a = \frac{I_1}{2}\left(1 + \frac{GI_1}{a}\right) + \frac{J_1}{2}\left(1 - \frac{GJ_1}{b}\right) + \frac{I_2}{2}\left(1 + \frac{GI_2}{OI_2}\right)$$
$$+ \frac{J_2}{2}\left(1 - \frac{GJ_2}{OJ_2}\right) + \frac{I_3}{2}\left(1 \pm \frac{GI_3}{OI_3}\right),$$

where the sign in the last term depends on which side of G the point I_3 is situated. If $b > a$, as in the figure, then the negative sign is to be taken. Here $c^2 = a^2 + b^2 + ab$, also $GJ_1^2 - GI_1^2 = b^2 - a^2$, and $GJ_1 + GI_1 = c$; whence

$$GI_1 = \frac{a^2}{c} + \frac{ab}{2c}, \quad GJ_1 = \frac{b^2}{c} + \frac{ab}{2c},$$

$$GI_2 = GI_1 - I_2I_1 = \frac{ab}{2c} = GJ_2, \qquad GI_3 = I_3I_1 - GI_1 = \frac{ab}{2c}\,\frac{b - a}{a + b},$$

$$OG^2 = \frac{3a^2b^2}{4c^2}, \quad OI_2 = \frac{ab}{c} = OJ_2, \quad OI_3 = \frac{ab}{a + b}.$$

Substituting for I_1, J_1, &c., GI_1, &c., in the expression given above, we obtain

$$E_a = \frac{L}{2}\left\{a + b + \frac{ab}{a + b} - \frac{2ab}{c} + \frac{a - b}{c}\left(a + b + \frac{ab}{2(a + b)}\right)\right\}$$

The total charge E on the conductor is given by the equation

$$E = E_a + E_b = L\left\{a + b + \frac{ab}{a + b} - \frac{2ab}{c}\right\}.$$

This result appears also from the consideration that $E = I_1 + J_1 + I_2 + J_2 + I_3$.

The above expressions for E_a, &c., hold good, whatever be the magnitudes of a and b.

120. **Uniplanar Inversion.**—In the case of uniplanar inversion corresponding charges are equal; and if E be the total amount of mass belonging to either of the inverse distributions, we have

$$V'_{P'} = \Sigma e' \log \frac{1}{P'A'} = \Sigma e \log \frac{OA}{PA \,.\, OP'} = V_P - V_0 + E \log \frac{1}{OP'}, \tag{25}$$

where V_0 denotes the potential at the origin of the distribution E.

It follows from this equation that, if the potential due to a uniplanar distribution of mass acting inversely as the distance be constant at all points of a curve s, and if the total mass be zero, the potential due to the inverse distribution is constant for all points of the inverse curve s', and the total mass is zero. Hence, if two charged points A_1 and A_2 be images of each other with respect to the curve s, the inverse points A'_1 and A'_2 are images of each other with respect to the curve s' which is the inverse of s.

Again, if the potential due to the uniplanar distribution of mass whose total amount is E be constant for all points of a curve s, the potential U' due to the inverse distribution together with a mass $-E$ placed at O, is constant for the curve s', which is the inverse of s, and the total mass producing the potential U' is zero.

If τ denote the density at any point, whose distance from O is r, of an areal distribution of uniplanar mass, and υ that of a linear distribution, and τ' and υ' the corresponding densities in the inverse distribution, remembering that in this case corresponding charges in the two systems are equal, we can prove in the same manner as in 3° and 4°, Art. 119, that

$$\tau' = \left(\frac{r}{R}\right)^4 \tau = \left(\frac{R}{r'}\right)^4 \tau, \quad \upsilon' = \left(\frac{r}{R}\right)^2 \upsilon = \left(\frac{R}{r'}\right)^2 \upsilon. \tag{26}$$

Examples.

1. Show that a circle c, throughout which there is a uniplanar distribution of mass m whose areal density varies inversely as the fourth power of the distance from an external point O, is centrobaric.

Invert from O, then

$$\tau' = \left(\frac{r}{R}\right)^4 \tau = \frac{K}{R^4}.$$

Hence the circle c', which is the inverse of c, being of uniform density, acts at external points as if its mass were concentrated at its centre A', and the potential U' due to c' together with $-m$ at A' is zero outside c', and therefore, at O, also the total mass producing U' is zero. Hence, by (25), if P be any point outside c, we have $U_P = 0$, and therefore, the potential in external space due to c is the same as that due to a mass m placed at A, the image of O in c.

2. Uniplanar mass m is distributed in the region outside a circle c, the areal density varying inversely as the fourth power of the distance from a point O inside c. Show that m acts inside c as if it were concentrated at A which is the image of O in c.

Invert from O, and we obtain a circle c' of uniform density. If U' be the potential due to c' together with a mass $-m$ placed at its centre A', the potential U' is zero at any point P' outside c'. Hence, since the total mass producing U' is zero, by (25), we have U constant throughout the region inside c, and therefore, m at A produces the same effect inside c as the original distribution outside c.

3. Uniplanar mass is distributed over the boundary of the larger segments of two circles cutting at an angle of 60° so as to produce a constant potential L throughout the interior region; find the potential at any external point, and the distribution of mass.

Let the spherical sections in the figure of Ex. 5, Art. 119 represent the circles; then if a charge η be placed at the points I_1, J_1, and I_3, and a charge $-\eta$ at I_2 and J_2, the potential produced is constant at the circular boundary, and if it be equal to L the charges at I_1, &c. produce in external space the same potential as the actual distribution. Hence η is determined by the equation

$$L = \eta\left\{\log\frac{a}{I_1J_1} - \log\frac{a}{I_1J_2} - \log a\right\} = \eta\log\frac{a+b}{c^2}.$$

If V be the potential at any external point Q, we have

$$\begin{aligned} V &= \eta\{\log I_2Q + \log J_2Q - \log I_1Q - \log J_1Q - \log I_3Q\} \\ &= \eta\log\frac{I_2Q\,.\,J_2Q}{I_1Q\,.\,J_1Q\,.\,I_3Q}. \end{aligned}$$

The density v of the distribution at any point P on the circle whose radius is a is given by the equation

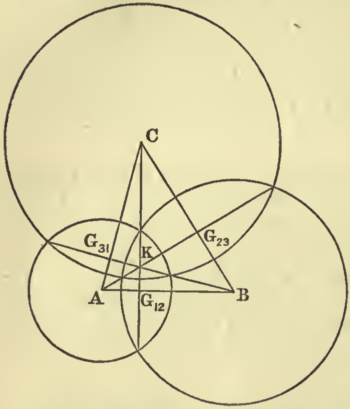
$$2\pi a v = \eta\left\{1 + \frac{a^2 - c^2}{J_1P^2} - \frac{a^2 - I_1J_2^2}{J_2P^2}\right\},$$

and the total charge E_a on the arc of this circle by the equation

$$2\pi E_a = \eta\{\alpha_2 + \alpha_3 + \beta_1 - \alpha_1 - \beta_2\},$$

where α_1, α_2 α_3, β_1, β_2, are the angles which the chord of intersection of the circles subtends at the points I_1, I_2, I_3, J_1, J_2, respectively.

121. Three Spheres cutting Orthogonally.—If three spheres cut orthogonally, a plane through their centres A, B, C meets them in circles cutting orthogonally, so that the centre of each lies on the common chord of the other two, and these chords AG_{23}, BG_{31}, and CG_{12}, are the three perpendiculars of the triangle ABC, and meet in a point K such that



$$AK \, . \, AG_{23} = AG_{31} \, . \, AC;$$

whence K is the image of G_{23} in the sphere whose centre is A and which may be denoted by (A). Similarly K is the image of G_{31} in (B) and of G_{12} in (C).

A potential L is produced at the surface of (A) by a charge aL at A, where a denotes the radius of A. Again, if the radii of (B) and (C) be denoted by b and c, and the distances AB, BC, and CA by γ, α, and β, charges bL at B and $\dfrac{-abL}{\gamma}$ at G_{12} produce on (A) a potential zero, as also charges cL at C and $\dfrac{-acL}{\beta}$ at G_{31}, finally to compensate the charge $\dfrac{-bcL}{\alpha}$ at G_{23}, there must be a charge

$$\frac{bcL}{\alpha}\frac{a}{AG_{23}} \text{ or } \frac{abcL}{2\Sigma} \text{ at } K,$$

where Σ is the area of the triangle ABC. If we express the sides and area of this triangle in terms of a, b, and c, we find that a potential L is produced at each of the spherical surfaces by charges

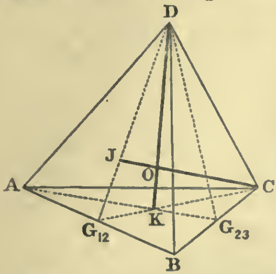
$$La, \quad Lb, \quad Lc, \quad \frac{-Lbc}{\sqrt{(b^2+c^2)}}, \quad \frac{-Lca}{\sqrt{(c^2+a^2)}}, \quad \frac{-Lab}{\sqrt{(a^2+b^2)}},$$

and

$$\frac{Labc}{\sqrt{(a^2b^2 + b^2c^2 + c^2a^2)}}$$

placed at A, B, C, G_{23}, G_{31}, G_{12}, and K, respectively.

122. **Four Spheres cutting Orthogonally.**—If a sphere cut two spheres, whose centres are A and B, orthogonally, its centre lies on the plane (AB) of the intersection of the spheres (A) and (B), and this plane is perpendicular to the line AB. Hence, if four spheres, whose centres are A, B, C, D, cut orthogonally, D lies on the line of intersection of the planes (AB) and (BC), which is perpendicular to the plane ABC, and passes through K the point of intersection of the perpendiculars of the triangle ABC. If G_{12} be the point in which AB meets the plane (AB), and J the foot of the perpendicular from C on the plane ABD, the lines DK and CJ are perpendiculars of the triangle DCG_{12}, and if O be their point of intersection, $CO \,.\, CJ = CK \,.\, CG_{12} = c^2$, where c is the radius of (C). In like manner AH, the perpendicular from A on the plane BCD, intersects CJ at the point O. Accordingly the four perpendiculars AH, BI, CJ, and DK intersect at the point O, and O is the image of H in (A), of I in (B), of J in (C), and of K in (D). Again, if Σ_1 denote the area of the triangle BCD, and Ω the volume of the tetrahedron $ABCD$, the image in (A) of the charge



$$\frac{Lbcd}{2\Sigma_1} \text{ at } H \text{ is } \frac{-Lbcd}{2\Sigma_1}\frac{a}{AH}, \text{ that is, } \frac{-Labcd}{6\Omega}.$$

By Art. 121, we have

$$CG_{12} \,.\, AB = 2\Sigma_4 = \sqrt{(a^2b^2 + b^2c^2 + c^2a^2)}.$$

Hence we obtain CG_{12} and CK, from which and DC we get DK, and finally we have

$$6\Omega = \sqrt{(a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2)}.$$

Thus all the charges are expressed in terms of L and the radii of the spheres, and we see that a potential L is produced at each of the spherical surfaces by placing charges La, &c., at the four centres,

$$\frac{-Lab}{\sqrt{a^2+b^2}}, \text{ \&c., at the six points } G_{12}, \text{ \&c.}$$

$$\frac{Labc}{\sqrt{(a^2b^2+b^2c^2+c^2a^2)}}, \text{ \&c., at the four points } K, \text{ \&c.,}$$

and $$\frac{-Labcd}{\sqrt{(a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2)}} \text{ at } O.$$

Examples.

1. An insulated conductor formed of the larger segments of three spheres cutting orthogonally is charged to potential L; find the density σ of the distribution at any point P on the surface of the sphere (A).

$$Ans.\ 4\pi\sigma = \frac{L}{a}\left\{1 - \frac{b^3}{BP^3} - \frac{c^3}{CP^3} + \frac{b^3c^3}{(b^2+c^2)^{\frac{3}{2}}\,G_{23}P^3}\right\}.$$

2. Find the density σ at a point P on (A) when the conductor charged to potential L is formed of the segments of four spheres cutting orthogonally.

$$Ans.\ 4\pi\sigma = \frac{L}{a}\left\{1 - \frac{b^3}{BP^3} - \frac{c^3}{CP^3} - \frac{d^3}{DP^3} + \frac{b^3c^3}{(b^2+c^2)^{\frac{3}{2}}\,G_{23}P^3} + \frac{c^3d^3}{(c^2+d^2)^{\frac{3}{2}}\,G_{34}P^3}\right.$$
$$\left. + \frac{d^3b^3}{(d^2+b^2)^{\frac{3}{2}}\,G_{24}P^3} - \frac{b^3c^3d^3}{(b^2c^2+c^2d^2+d^2b^2)^{\frac{3}{2}}\,HP^3}\right\}.$$

3. A conductor, formed of the segments of three spheres cutting orthogonally, and having the centres of the spheres in its interior, is at potential zero under the influence of an external electrified point P; show how to determine the distribution of mass on the surface of the conductor, and the potential in external space.

Let the spheres be denoted by (A), (B), and (C). The plane of the intersection of (A) and (B) cuts (C) in a circle which meets the circle of intersection of (A) and (B) in two points O and O_2. Invert the system from one of these points O, then the inverses of the spheres are planes, (A'), (B'), (C') intersecting perpendicularly at the point O'_2, and the inverse of P is a point P' in the region bounded by the quadrants of the three planes. Let ξ, η, ζ be the perpendicular distances of P' from the planes, then

$$\xi = \frac{R^2}{2a}, \qquad \eta = \frac{R^2}{2b}, \qquad \zeta = \frac{R^2}{2c}.$$

By changing the sign of the coordinate of a point, we obtain the coordinates of the image point in the corresponding coordinate plane. Thus in the present case, the whole system is found by taking all possible combinations of algebraical signs prefixed to the coordinates ξ, η, ζ. The sign of the charge to be placed at any point is negative if it has an odd number of negative coordinates, and positive if it has an even number. If the image of P' in the plane (A') be denoted by P'_a, with a corresponding notation for the other points, we have the system:—

Points P', P'_a, P'_b, P'_c, P'_{ab}, P'_{ac}, P'_{bc}, P'_{abc};

Coordinates, $\xi\eta\zeta$, $-\xi\eta\zeta$, $\xi-\eta\zeta$, $\xi\eta-\zeta$, $-\xi-\eta\zeta$, $-\xi\eta-\zeta$, $\xi-\eta-\zeta$, $-\xi-\eta-\zeta$;

Charges, e', $-e'$, $-e'$, $-e'$, e', e', e', $-e'$.

In the inverse spherical system, since O is a point common to the three spheres, if we put $e=\mu OP$, we have for the electrified points, which produce in external space the actual potential:—

Points, P, P_a, P_b, P_c, P_{ab}, P_{ac}, P_{bc}, P_{abc};

Charges, μOP, $-\mu OP_a$, $-\mu OP_b$, $-\mu OP_c$, μOP_{ab}, μOP_{ac}, μOP_{bc}, $-\mu OP_{abc}$,

where P_a is the image of P in the sphere (A), P_{ab} the image of P_a in (B), or of P_b in (A), &c.

Since the resultant force in the interior of the conductor is zero, the density of the distribution at any point of its surface can be determined as in preceding Examples.

4. A conductor, formed of the segments of four spheres cutting orthogonally and having their centres in its interior, is at potential zero under the influence of an external electrified point P; show how to determine the distribution of mass on the conductor, and the potential in external space.

Adopting the same notation as that of the last Example and calling the fourth sphere (D), if we invert from O, one of the points common to (A), (B), and (C), we get three rectangular planes and a sphere (D') cutting them perpendicularly, and having, therefore, O'_2 as its centre. If P' be the inverse of P, the successive images of P' in the planes form the same system P'_a, &c., as that considered in the last Example. These points lie on a sphere having O'_2 for centre; and their images in the concentric sphere (D') are equidistant from O'_2, and are obviously in reference to each other a complete system of images in the planes (A'), (B'), and (C'). Hence, by 5°, Art. 119, in the inverse system of four spheres, a potential zero is obtained at the surface of each by placing at the points

$$P,\ P_a,\ P_b,\ P_c,\ P_{ab},\ P_{ac},\ P_{bc},\ P_{abc},\ P_d,\ P_{ad},\ P_{bd},\ P_{cd},\ P_{abd},\ P_{acd},\ P_{bcd},\ P_{abcd},$$

charges

$$\mu OP,\ -\mu OP_a,\ -\mu OP_b,\ -\mu OP_c,\ \mu OP_{ab},\ \mu OP_{ac},\ \mu OP_{bc},\ -\mu OP_{abc},\ -\nu OP_d,$$
$$\nu OP_{ad},\ \nu OP_{bd},\ \nu OP_{cd},\ -\nu OP_{abd},\ -\nu OP_{acd},\ -\nu OP_{bcd},\ \nu OP_{abcd},$$

where

$$\mu OP = e, \text{ and } \nu OP_d = \frac{d}{DP}e.$$

123. **Finite Series of Images and continuous Inverse Systems**.—The relation between systems of images and the surface distributions to which they are equivalent may usually be based on the following general theorem:—

If a closed surface S be one of equilibrium for a distribution of mass of which part is external to S and part internal, a surface distribution on S, whose density at any point is equal to the resultant force at that point divided by 4π, produces at all points external to S a potential equal to that of the internal mass, and at all internal points a potential whose difference from a constant is equal to the potential of the external mass.

This theorem is easily proved by supposing a distribution on S whose potential at all points of this surface is equal to that of the internal mass.

Numerous problems of considerable difficulty have been solved by Thomson, Clerk Maxwell, and other mathematicians, by the use of images finite in number, or by the inversion of one continuous system into another. Some of the most important of these problems are given in the following Examples:—

Examples.

1. An insulated conductor formed of the larger segments of two spheres cutting at an angle $\frac{\pi}{n}$, where n is any integer, is charged to potential L; find the distribution of mass on the surface of the conductor, and the potential in external space.

Adopting the method and the notation of Ex. 5, Art. 119, we find

$$i'_{2m} = 2m(\alpha + \beta), \qquad j'_{2m} = -2m(\alpha + \beta),$$

$$i'_{2m+1} = 2\alpha + 2m(\alpha + \beta), \quad j'_{2m+1} = -2\beta - 2m(\alpha + \beta).$$

In this case $\alpha + \beta = \frac{\pi}{n}$, and therefore, whether $n = 2m$ or $2m + 1$, we have

$$i'_n - j'_n = 2n(\alpha + \beta) = 2\pi,$$

and the point I'_n coincides with J'_n.

In the inverse spherical system, I_1 and J_1 being the centres of the spheres (A) and (B), we have the images I_2, I_3, &c., I_n in the sphere (A), and the images J_2, J_3, &c., J_n in the sphere (B), and J_n coincides with I_n. The charges to be placed at these points, the distribution of mass on the conductor, and the

potential in external space are found then as in Ex. 5, Art. 119. The total mass E_a on the spherical surface whose radius is a is given by the equation

$$E_a = \frac{L}{2}\{OI_1 + OJ_1 + OI_3 + OJ_3 + \&c. + GI_1 + GJ_1 + GI_3 + GJ_3 + \&c.$$
$$- OI_2 - OJ_2 - OI_4 - OJ_4 - \&c. - GI_2 - GJ_2 - GI_4 - GJ_4 - \&c.\},$$

where GJ_1, GI_2, &c. are to be taken as positive or negative according as the points J_1, I_2, &c. are on the same side of G as I_1 or on the opposite side.

2. A spherical bowl is at potential zero under the influence of an external electrified point O situated on the surface of the sphere (A) in space of which geometrically the bowl forms a part; find the distribution of mass on the surface of the bowl.

Invert from O, the sphere (A) becomes a plane, and the plane base of the bowl a sphere, so that the surface of the bowl is inverted into a plane circular disk. If we suppose this disk at constant potential L', by Ex. 7, Art. 38, the density σ' of the distribution at any point P' of the disk is given by the equation

$$\sigma' = \frac{k}{\sqrt{(a'^2 - r'^2)}},$$

where k is a constant, a' is the radius of the disk, and r' the distance of P' from its centre; then, if S' be one surface of the disk,

$$L' = 2\int \frac{\sigma' dS'}{r'} = 4\pi k \int_0^{a'} \frac{dr'}{\sqrt{(a'^2 - r'^2)}} = 2\pi^2 k;$$

whence

$$\sigma' = \frac{L'}{2\pi^2}\frac{1}{\sqrt{H'P' \cdot P'K'}},$$

where H' and K' are the extremities of any chord of the disk passing through P'.

If H and K be the points on the edge of the bowl inverse to H' and K', we have

$$\frac{HP}{H'P'} = \frac{OP}{OH'} = \frac{OP \cdot OH}{R^2}, \quad \frac{PK}{P'K'} = \frac{OP}{OK'} = \frac{OP \cdot OK}{R^2}; \text{ also } \sigma = \sigma'\left(\frac{R}{OP}\right)^3;$$

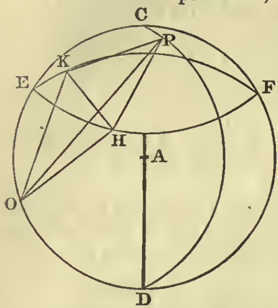
whence

$$\sigma = \frac{L'}{2\pi^2}\frac{R}{OP^2}\sqrt{\frac{OH \cdot OK}{PH \cdot PK}}.$$

If e be the charge at O, in order that the bowl should be at potential zero by 2°, (b), Art 119, we have $L'R = -e$, and therefore,

$$\sigma = \frac{-e}{2\pi^2}\frac{1}{OP^2}\sqrt{\frac{OH \cdot OK}{PH \cdot PK}}.$$

Since the three points H', P', K' are on the same straight line, the four points O, H, P, K are on the circumference of a circle whose plane cuts the base of the bowl in the line HK at an angle θ; then, p and q being the perpendiculars from O and P on the line HK, we have



$$\frac{OH \cdot OK}{PH \cdot PK} = \frac{p}{q} = \frac{p \sin\theta}{q \sin\theta} = \frac{\zeta_0}{\zeta_p},$$

where ζ_0 and ζ_p are the perpendiculars from O and P on the base of the bowl. If C denote the pole of the small circle forming the edge of the bowl, b the distance from C of any point of this edge, z the coordinate of any point of the sphere (A) referred to C as origin and the diameter through C as axis, and f the length of the diameter, we have

$$\zeta_0 = z_0 - \frac{b^2}{f} = \frac{CO^2 - b^2}{f}, \quad \zeta_p = \frac{b^2}{f} - z_p = \frac{b^2 - CP^2}{f};$$

whence

$$\sigma = \frac{-e}{2\pi^2 OP^2}\sqrt{\frac{\zeta_0}{\zeta_r}} = \frac{-e}{2\pi^2 OP^2}\sqrt{\left(\frac{CO^2 - b^2}{b^2 - CP^2}\right)}.$$

The value of σ here obtained may perhaps be more readily deduced from that of σ' by considering the sphere (B') which is the inverse of the base of the bowl.

If Q be the point in which OP meets the base of the bowl, and Q' the inverse point on (B'), since $H'P'K'$ is a chord of (B'), we have

$$H'P' \cdot P'K' = OP' \cdot P'Q', \quad \text{but} \quad P'Q' = PQ\,\frac{OQ'}{OP};$$

whence

$$H'P' \cdot P'K' = \frac{OP'}{OP}\,PQ \cdot OQ' = \frac{R^4}{OP^2}\frac{PQ}{OQ} = \frac{R^4}{OP^2}\frac{\zeta_P}{\zeta_0},$$

and from this the value of σ follows immediately.

3. An insulated spherical bowl is charged to constant potential L; find the distribution of mass on its surface.

Imagine a distribution of mass of uniform density ϵ on the remainder of the sphere (A) of which the bowl is part. If the bowl be at potential zero under the influence of this distribution, there will be a distribution on the surface of the bowl whose density at any point P may be denoted by σ. Suppose, further, an additional distribution of uniform density ϵ' over the whole sphere (A), then the bowl is at a constant potential, and the density σ_1 of the distribution at any point of its outer surface is $\sigma + \epsilon'$, and at any point of its inner is σ.

Let us now assume that $\epsilon + \epsilon' = 0$, and that $4\pi a\epsilon' = L$, where a is the radius of (A), and we obtain the distribution on an uninfluenced bowl at potential L.

To calculate σ, let Q denote any point of the sphere (A) beyond the edge of the bowl, A the centre of the sphere, C the centre of the bowl or the pole of its

boundary, θ and ϕ the spherical coordinates of Q referred to AC and the plane ACP, c and p the perpendicular distances of A and P from the base of the bowl, β the angle subtended by the radius of this base at A, and α the angle PAC; then by Ex. 2,

$$-d\sigma = \frac{\epsilon a^2 \sin\theta d\theta d\phi}{2\pi^2 PQ^2}\sqrt{\left(\frac{c - a\cos\theta}{p}\right)},$$

also

$$PQ^2 = 2a^2\{1 - (\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\phi)\};$$

whence

$$-\sigma = \frac{\epsilon}{2\pi^2\sqrt{p}}\int_\beta^\pi\int_0^\pi \frac{\sqrt{(c - a\cos\theta)}\sin\theta d\theta d\phi}{1 - \cos\alpha\cos\theta - \sin\alpha\sin\theta\cos\phi}.$$

The integral with respect to ϕ is of the form

$$\int_0^\pi \frac{d\phi}{m - n\cos\phi}, \text{ the value of which is } \frac{\pi}{\sqrt{(m^2 - n^2)}}.$$

In the present case $m^2 - n^2 = \cos^2\alpha + \cos^2\theta - 2\cos\alpha\cos\theta$; hence putting $\cos\theta = \mu$, we have

$$-\sigma = \frac{\epsilon}{2\pi\sqrt{p}}\int_{-1}^{\mu_1}\frac{\sqrt{(c - a\mu)}\, d\mu}{\cos\alpha - \mu}, \text{ where } \mu_1 = \cos\beta.$$

Remembering that $c = a\mu_1$, and that $a\cos\alpha - c = p$, we find by integrating that

$$-\sigma = \frac{\epsilon}{\pi}\left\{\sqrt{\left(\frac{a+c}{p}\right)} - \tan^{-1}\sqrt{\left(\frac{a+c}{p}\right)}\right\}.$$

If we denote by b the distance of C from any point on the edge of the bowl, and if we put $2a = f$, $CP = r$, we have

$$a + c = \frac{f^2 - b^2}{f}, \quad p = \frac{b^2}{f} - \frac{r^2}{f},$$

whence

$$-\sigma = \frac{\epsilon}{\pi}\left\{\sqrt{\left(\frac{f^2 - b^2}{b^2 - r^2}\right)} - \tan^{-1}\sqrt{\left(\frac{f^2 - b^2}{b^2 - r^2}\right)}\right\}.$$

Substituting for ϵ its value $\frac{-L}{4\pi a}$, we get, for σ_1 and σ_2, the densities on the outer and inner surfaces of the bowl, the equations

$$\sigma_1 = \epsilon' + \sigma = \frac{L}{2\pi f}\left\{1 + \frac{1}{\pi}\left[\sqrt{\left(\frac{f^2 - b^2}{b^2 - r^2}\right)} - \tan^{-1}\sqrt{\left(\frac{f^2 - b^2}{b^2 - r^2}\right)}\right]\right\},$$

$$\sigma_2 = \sigma = \frac{L}{2\pi^2 f}\left\{\sqrt{\left(\frac{f^2 - b^2}{b^2 - r^2}\right)} - \tan^{-1}\sqrt{\left(\frac{f^2 - b^2}{b^2 - r^2}\right)}\right\}.$$

If the point of the sphere (A), which is opposite to C, be denoted by D, and ζ_D and ζ_P denote the perpendiculars from D and P on the base of the bowl, it

is plain that $a + c = \zeta_D$, $p = \zeta_P$, and that σ may be expressed by the equation

$$\sigma = \frac{L}{2\pi^2 f}\left\{\sqrt{\left(\frac{\zeta_D}{\zeta_P}\right)} - \tan^{-1}\sqrt{\left(\frac{\zeta_D}{\zeta_P}\right)}\right\}.$$

4. Find the distribution of mass on a spherical bowl (B) at potential zero under the influence of a charge e situated at any point O outside the bowl.

Invert the system from O, selecting the radius of inversion R so that the sphere (A), of which the bowl is part, may be inverted into itself. The bowl (B) is inverted then into another portion of (A) bounded by the intersection of (A) with the sphere which is the inverse of the base of (B), that is, (B) is inverted into another bowl (B') belonging to the same sphere.

If (B') be at constant potential L' due to a distribution on (B'), by the last Example the density σ' of the distribution at any point P' of the inner surface of (B') is given by the equation

$$\sigma' = \frac{L'}{2\pi^2 f}\left\{\sqrt{\left(\frac{\zeta'_{D'}}{\zeta'_{P'}}\right)} - \tan^{-1}\sqrt{\left(\frac{\zeta'_{D'}}{\zeta'_{P'}}\right)}\right\}.$$

Draw a plane through $D'P'$ meeting the edge of (B') in the points H', K'; then the points D', H', P', K' being concyclic, so also are the inverse points D, H, P, K; and we have, as in Ex. 2,

$$\frac{\zeta'_{D'}}{\zeta'_{P'}} = \frac{D'H' \cdot D'K'}{P'H' \cdot P'K'}, \qquad \frac{\zeta_D}{\zeta_P} = \frac{DH \cdot DK}{PH \cdot PK},$$

where ζ_D and ζ_P are perpendiculars from D and P on the base of (B).

Again, by similar triangles, as in 2°, Art. 119,

$$D'H' = \frac{OD'}{OH} DH, \quad D'K' = \frac{OD'}{OK} DK,$$

$$P'H' = \frac{OP'}{OH} PH, \quad P'K' = \frac{OP'}{OK} PK,$$

and therefore,

$$\frac{D'H' \cdot D'K'}{P'H' \cdot P'K'} = \frac{OD'^2}{OP'^2}\frac{DH \cdot DK}{PH \cdot PK} = \frac{OP^2}{OD^2}\frac{DH \cdot DK}{PH \cdot PK}.$$

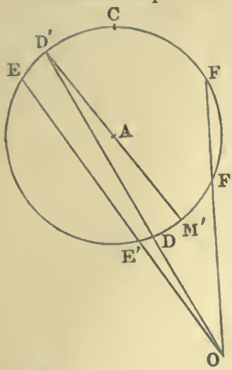
If σ be the density, corresponding to σ', of the distribution on (B), by (22), Art. 119, we have

$$\sigma = \left(\frac{R}{OP}\right)^3 \sigma',$$

and (B) is at potential zero under the influence of a charge e at O, provided that $RL' = -e$.

Hence, substituting in σ', we get

$$\sigma = \frac{-e}{2\pi^2 f}\frac{R^2}{OP^3}\left\{\frac{OP}{OD}\sqrt{\left(\frac{\zeta_D}{\zeta_P}\right)} - \tan^{-1}\frac{OP}{OD}\sqrt{\left(\frac{\zeta_D}{\zeta_P}\right)}\right\}.$$



D is here the point in which OD' meets the sphere (A) again, and D is found by taking the points E, F in which the plane OAC meets the edge of (B) and joining them to O, the joining lines meet (A) again in the points E', F' on the edge of (B'); then D' is the point of (A) opposite to M', the middle point of the arc $E'\ F'$ of the circle in which AOC meets (A).

To prove the validity of this construction it is only necessary to show that the pole of the circle which is the boundary of (B'), lies in the plane AOC, which appears thus. AC is perpendicular to the base of (B), and therefore a perpendicular to this base from O, which passes through the centre of the inverse sphere, lies in the plane AOC. Hence the line joining the centres of the two spheres whose intersection is the boundary of (B') lies in the plane AOC, but the pole of the boundary is on this line and therefore in the plane AOC.

As in the preceding Examples σ may be put into the form

$$\sigma = \frac{-e}{2\pi^2 f}\frac{R^2}{OP^3}\left\{\frac{OP}{OD}\sqrt{\left(\frac{CD^2-b^2}{b^2-CP^2}\right)} - \tan^{-1}\frac{OP}{OD}\sqrt{\left(\frac{CD^2-b^2}{b^2-CP^2}\right)}\right\}.$$

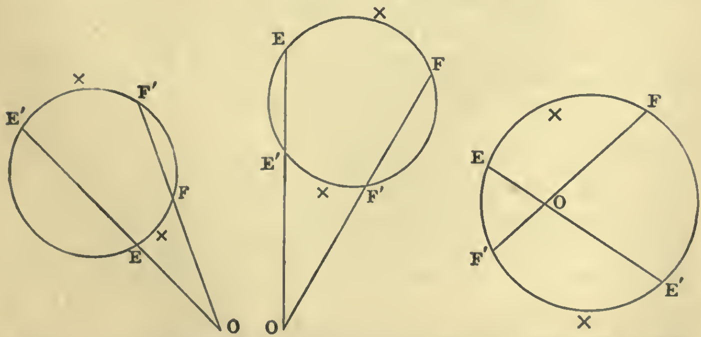
On the outside of (B') the density is

$$\sigma' + \frac{L'}{2\pi f}, \text{ that is, } \sigma' - \frac{e}{2\pi f R}.$$

The corresponding density σ_1 on (B) is given by the equation

$$\sigma_1 = \sigma - \frac{e}{2\pi f}\frac{R^2}{OP^3}.$$

σ_1 is the density of the distribution on that surface of (B) which can be reached from O without passing through the surface of (A). This appears by consider-



ing the three possible cases, and by remembering that the surface of (B) which is next to O corresponds to the surface of (B') which is most remote.

5. A circular disk is at potential zero under the influence of a charge e situated at an external point O in its plane; find the distributiou of mass on the disk.

In Ex. 2, if we suppose the diameter of the sphere (A) to become infinite, and b to remain finite, the bowl (B) becomes a disk whose centre is C, and whose radius is b, and we get for the density σ of the distribution at any point P of the disk the equation

$$\sigma = \frac{-e}{2\pi^2 OP^2}\sqrt{\left(\frac{CO^2 - b^2}{b^2 - CP^2}\right)}.$$

6. Give a direct construction for the point D in Ex. 4.

The point D' is equally distant from all points of the edge of the bowl (B'), but if H' be a point on this edge, and H the inverse point on the edge of (B), we have $\frac{D'H'}{OD'} = \frac{DH}{OH}$. Hence, if H be any point on the edge of (B), the ratio $\frac{DH}{OH}$ is constant; accordingly $\frac{DE}{DF} = \frac{OE}{OF}$, and therefore the locus of D is a circle, whose intersection with the circle ECF determines two points, of which D is the one not on the surface of the bowl (B).

124. **Spheres in Contact.**—If two spheres (A) and (B), whose radii are a and b, and which touch at the point O, be inverted from this point, the inverse surfaces (A') and (B') are parallel planes whose distances from O are α and β, where $\alpha = \frac{R^2}{2a}$, $\beta = \frac{R^2}{2b}$. If these planes be at potential zero under the influence of a charge e' at O, the potential at any point between the planes is the same as that due to the charge e' at O, together with charges at the successive images of O, which form an infinite series.

The images in the plane (A') being denoted by I'_1, I'_2, &c., and those in the plane (B') by J'_1, J'_2, &c., the two sets are on opposite sides of O, and if $OI'_1 = \xi'_1$, $OI'_2 = \xi'_2$, &c., $OJ'_1 = \eta'_1$, $OJ'_2 = \eta'_2$, &c., and $\alpha + \beta = \gamma$, we have $\xi'_1 = 2\alpha$, $\eta'_1 = 2\beta$, $\xi'_2 = \alpha + \eta'_1 + \alpha = 2\gamma$, $\eta'_2 = \beta + \xi'_1 + \beta = 2\gamma$, $\xi'_3 = 2\alpha + 2\gamma$, $\eta'_3 = 2\beta + 2\gamma$, $\xi'_{2n} = 2n\gamma$, $\eta'_{2n} = 2n\gamma$, $\xi'_{2n-1} = 2\alpha + (2n-2)\gamma = 2n\gamma - 2\beta$, $\eta'_{2n-1} = 2n\gamma - 2\alpha$.

In the inverse system, I_1 and J_1 are the centres of the spheres (A) and (B), and in order that the spheres should be at potential L, by 2°, Art. 119, we have $e' = -LR$; whence the charge at I'_1 is LR, and that at I_1 is $\frac{LR^2}{OI'_1}$, that is, $\frac{LR^2}{2\alpha}$ or La. Similarly the charge at J_1 is Lb.

Since the charge at any image point is proportional to its distance from O, the charge at I_{2n} is $-L\xi_{2n}$, and that at I_{2n-1} is $L\xi_{2n-1}$, but

$$\xi_{2n} = \frac{R^2}{\xi'_{2n}} = \frac{R^2}{2n\left(\frac{R^2}{2a} + \frac{R^2}{2b}\right)} = \frac{ab}{n(a+b)},$$

and

$$\xi_{2n-1} = \frac{R^2}{2n\left(\frac{R^2}{2a} + \frac{R^2}{2b}\right) - \frac{R^2}{b}} = \frac{ab}{n(a+b) - a}.$$

If we put $\frac{b}{a+b} = \mu$, which gives $\frac{-a}{a+b} = \mu - 1$, we get for the sum E_a of the charges at the centre and succession of images belonging to the sphere (A)

$$E_a = L\frac{ab}{a+b}\sum_1^\infty \left(\frac{1}{n+\mu-1} - \frac{1}{n}\right). \tag{27}$$

As

$$\Sigma\frac{1}{n} = \int_0^1 \frac{d\theta}{1-\theta}, \text{ and } \Sigma\frac{1}{n+\mu-1} = \int_0^1 \frac{\theta^{\mu-1}d\theta}{1-\theta},$$

we may write

$$E_a = L\frac{ab}{a+b}\int_0^1 \frac{(\theta^{\mu-1} - 1)\,d\theta}{1-\theta}. \tag{28}$$

By putting $\frac{a}{b} = \nu$, we have, $\mu = \frac{1}{1+\nu}$, and

$$E_a = La\nu\sum_1^\infty \frac{1}{n^2}\left\{1 + \nu\left(2 - \frac{1}{n}\right) + \nu^2\left(1 - \frac{1}{n}\right)\right\}^{-1}. \tag{29}$$

If a be small compared with b, the higher powers of ν may be neglected, and we have $E_a = La\nu\Sigma\frac{1}{n^2}$. By comparing the coefficients of x^3 in the two expansions for $\sin x$ we get, as is well known,

$$\Sigma\frac{1}{n^2} = \frac{\pi^2}{6}. \quad \text{Hence} \quad E_a = \frac{\pi^2}{6}\frac{La^2}{b}. \tag{30}$$

Again

$$E_b = L\sum_1^\infty \left\{\frac{ab}{n(a+b)-b} - \frac{ab}{n(a+b)}\right\}$$

$$= Lb\sum_1^\infty \left\{\frac{\nu}{n(1+\nu)-1} - \frac{\nu}{n(1+\nu)}\right\}.$$

By making n unity, and then changing n into $n+1$, in the first term under the sign of summation, we get

$$E_b = Lb\nu\left\{\frac{1}{\nu} - \sum_1^\infty \left(\frac{1}{n+n\nu} - \frac{1}{n+(n+1)\nu}\right)\right\}$$

$$= Lb\nu\left\{\frac{1}{\nu} - \nu\Sigma\frac{1}{n^2}\left[1+\nu\left(2+\frac{1}{n}\right)+\nu^2\left(1+\frac{1}{n}\right)\right]^{-1}\right\}; \quad (31)$$

and if the higher powers of ν be neglected, we have

$$E_b = Lb\left(1 - \frac{\pi^2}{6}\frac{a^2}{b^2}\right) = Lb - E_a. \quad (32)$$

From (30) and (32) we get for the mean values σ_a and σ_b of the densities of the distributions on the two spheres

$$\sigma_a = \frac{\pi^2}{6}\frac{L}{4\pi b}, \qquad \sigma_b = \frac{L}{4\pi b}\left(1 - \frac{\pi^2}{6}\frac{a^2}{b^2}\right),$$

and therefore,

$$\sigma_a = \frac{\pi^2}{6}\sigma_b. \quad (33)$$

Equations (27) and (28) give expressions for the total charge on one of two spheres in contact which are insulated and charged to potential L.

Equations (29) and (31) give expressions for the total charge on each sphere when one is small compared with the other. Equation (33) is rigorously true for a sphere in contact with an infinite charged plane. If the spheres (A) and (B) be equal, $\mu = \frac{1}{2}$; and from (27), we have

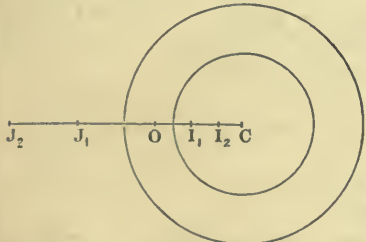
$$E_a = E_b = La\sum_1^\infty \left(\frac{1}{2n-1} - \frac{1}{2n}\right) = La\log_e 2 = 0{\cdot}693147La. \quad (34)$$

125. **Concentric Spheres.**—If two concentric spheres (A) and (B) be at potential zero under the influence of a charge e at a point O situated between them, the potential at any point between the spheres is that due to the charge at O, together with charges at the successive images of O in the spheres. If I_1, I_2, &c., denote the images in (A), ξ_1, ξ_2, &c., their distances from C the centre, i_1, i_2, &c., the charges at these points, J_1, &c., η_1, &c., and j_1, &c., the corresponding points and quantities for (B), denoting the radii of the spheres by a and b, and putting $CO = f$, and $a = \mu b$, the inner sphere being (A), we have

$$\xi_1 = \frac{a^2}{f}, \quad \eta_1 = \frac{b^2}{f}, \quad \xi_2 = \frac{a^2}{\eta_1} = \mu^2 f, \qquad \eta_2 = \mu^{-2} f.$$

Assuming then

$$\xi_{2n-1} = \mu^{2(n-1)}\frac{a^2}{f}, \; \xi_{2n} = \mu^{2n} f, \; \eta_{2n-1} = \mu^{-2(n-1)}\frac{b^2}{f}, \; \eta_{2n} = \mu^{-2n} f, \quad (35)$$



since $\xi_{n+1} = \dfrac{a^2}{\eta_n}$, and $\eta_{n+1} = \dfrac{b^2}{\xi_n}$, we see that, as the above assumptions hold good for $n = 1$, they hold good in general. Also, since by (1), Art. 110, the charges at a point and its image in a sphere are proportional to the square roots of their distances from the centre, we have

$$i_n = \pm\sqrt{\frac{\xi_n}{f}}\, e, \quad j_n = \pm\sqrt{\frac{\eta_n}{f}}\, e;$$

whence

$$i_{2n-1} = -\mu^{n-1}\frac{a}{f}\, e, \quad i_{2n} = \mu^n e, \; j_{2n-1} = -\mu^{-(n-1)}\frac{b}{f}\, e, \; j_{2n} = \mu^{-n} e. \quad (36)$$

Since the images in (A) produce the same potential in external space as the distribution on its surface, if E_a be the total charge on (A), we have

$$E_a = \Sigma i = e\left\{\Sigma\mu^n - \frac{a}{f}\,\Sigma\mu^{n-1}\right\} = \frac{e}{1-\mu}\left\{\mu - \frac{a}{f}\right\}.$$

Again, as the potential at (B) is zero, the total charge E_b on its surface is equal and opposite in algebraical sign to the sum of the interior masses; hence

$$E_b = -(e + E_a) = -\frac{e}{1-\mu}\left(1 - \frac{a}{f}\right).$$

The total charges E_a and E_b may be expressed by the equations

$$E_a = \frac{-e}{1-\mu}\,\frac{a}{b}\,\frac{b-f}{f}, \qquad E_b = \frac{-e}{1-\mu}\,\frac{f-a}{f};$$

hence, if A and B denote the points in which OC meets the spheres (A) and (B), we have

$$\frac{E_a}{E_b} = \frac{a}{b}\,\frac{OB}{OA}, \qquad E_a + E_b = -e.$$

If we put $b - a = c$, and suppose c to remain finite while a and b become infinite, the spheres become parallel planes, and we find for the total charges E_1 and E_2 on parallel planes at potential zero under the influence of a charge e situated at a point O between the planes, the equations

$$E_1 = -\frac{ep_2}{c}, \qquad E_2 = -\frac{ep_1}{c}, \tag{37}$$

where p_1 and p_2 are the distances of O from the planes, and c is the distance between them.

126. **Spheres influencing each other.**—If a sphere (B), at potential zero, be in the presence of an insulated sphere (A) at potential L, the total mass on each sphere can be expressed by an infinite series deducible by the method of images. This mode of obtaining the series was first employed by Thomson. The special form of investigation here adopted is due to Kirchhoff.

Let the centres of the spheres be denoted by A and B, their radii by a and b, and the distance AB by c.

A charge La placed at A produces a potential L at the surface of (A); but in order to have the potential zero at (B), a charge must be placed at J_0, the image of A in (B). To render the addition to the potential zero at (A) another charge must be supposed at I_1 the image of J_0 in (A), and so

on. Let the charges at A, I_1, I_2, &c., be denoted by i_0, i_1, i_2, &c., those at J_0, J_1, J_2, &c., by j_0, j_1, j_2, &c., the distances AI_1, AI_2, &c., by f_1, f_2, &c., and the distances BJ_0, BJ_1, BJ_2, &c., by h_0, h_1, h_2, &c., then

$$i_{n+1} = -\frac{f_{n+1}}{a} j_n, \quad j_n = -\frac{h_n}{b} i_n, \; f_{n+1} = \frac{a^2}{c - h_n}, \quad h_n = \frac{b^2}{c - f_n}. \qquad (38)$$

Eliminating f_{n+1}, j_n, and h_n from these equations, we get

$$i_{n+1} = \frac{ab}{c^2 - b^2 - cf_n} i_n.$$

If we eliminate j_n and h_n from the first three of equations (38), we obtain

$$i_{n+1} = \frac{cf_{n+1} - a^2}{ab} i_n, \text{ which gives } i_n = \frac{cf_n - a^2}{ab} i_{n-1},$$

from the two equations expressing i_{n+1} and i_{n-1} in terms of i_n and f_n, we have

$$\frac{i_n}{i_{n+1}} + \frac{i_n}{i_{n-1}} = \frac{c^2 - a^2 - b^2}{ab}. \qquad (39)$$

As this equation contains neither f nor h, and is symmetrical in a and b, we have also

$$\frac{j_n}{j_{n+1}} + \frac{j_n}{j_{n-1}} = \frac{c^2 - a^2 - b^2}{ab}.$$

If we put $\frac{1}{i_n} = a_n$, and assume $a_{n-1} = A\nu^{n-1} + B\nu^{-(n-1)}$, $a_n = A\nu^n + B\nu^{-n}$,

$$\nu + \frac{1}{\nu} = \frac{c^2 - a^2 - b^2}{ab}, \qquad (40)$$

we get from (39), $a_{n+1} = A\nu^{n+1} + B\nu^{-(n+1)}$; hence the equation

$$a_n = \frac{1}{i_n} = A\nu^n + B\nu^{-n}, \qquad (41)$$

where ν and $\frac{1}{\nu}$ are the roots of the equation

$$x^2 - \frac{c^2 - a^2 - b^2}{ab} x + 1 = 0,$$

holds good for all values of n.

In like manner we obtain

$$\beta_n = \frac{1}{j_n} = C\nu^n + D\nu^{-n}. \tag{42}$$

A and B are determined from the values of i_0 and i_1, and C and D from those of j_0 and j_1. Since $i_0 = La$, we have

$$j_0 = -\frac{b}{c}\,i_0 = -\frac{Lab}{c}, \quad i_1 = \frac{-a}{c - \dfrac{b^2}{c}}\,j_0 = \frac{La^2b}{c^2 - b^2},$$

$$j_1 = \frac{-b}{c - \dfrac{a^2}{c - \dfrac{b^2}{c}}}\,i_1 = \frac{-b(c^2 - b^2)}{c(c^2 - a^2 - b^2)}\,\frac{La^2b}{c^2 - b^2} = \frac{-La^2b^2}{c(c^2 - a^2 - b^2)}.$$

Hence we obtain

$$A + B = a_0 = \frac{1}{La}, \quad A\nu + B\nu^{-1} = a_1 = \frac{c^2 - b^2}{La^2b}, \tag{43}$$

$$C + D = \beta_0 = -\frac{c}{Lab}, \quad C\nu + D\nu^{-1} = \beta_1 = -\frac{c(c^2 - a^2 - b^2)}{La^2b^2}. \tag{44}$$

Dividing the second of equations (43) by the first, and putting $\dfrac{A}{B} = \xi$, we have

$$\frac{\xi\nu + \nu^{-1}}{\xi + 1} = \frac{c^2 - b^2}{ab} = \nu + \nu^{-1} + \frac{a^2}{ab}.$$

Solving for ξ, we obtain

$$\xi = -\frac{\nu(a + \nu b)}{b + \nu a} = -\frac{\nu(a + \nu b)^2}{ab\nu^2 + (a^2 + b^2)\,\nu + ab} = -\frac{\nu(a + \nu b)^2}{\nu c^2};$$

hence

$$\frac{A}{B} = -\lambda^2, \text{ where } \lambda = \frac{a + \nu b}{c}. \tag{45}$$

Taking ν as that root of the equation for x which is less than unity, we have $a + \nu b < c$, and $\lambda < 1$, then

$$A = (A + B)\,\frac{\lambda^2}{\lambda^2 - 1}, \quad B = \frac{A + B}{1 - \lambda^2},$$

and

$$i_n = \frac{1}{a_n} = \frac{1}{A\nu^n + B\nu^{-n}} = \frac{La\nu^n(1-\lambda^2)}{1-\lambda^2\nu^{2n}}. \qquad (46)$$

If we divide the second of equations (44) by the first, and put $\frac{C}{D} = \eta$, we get

$$\frac{\eta\nu + \nu^{-1}}{\eta + 1} = \frac{c^2 - a^2 - b^2}{ab} = \nu + \nu^{-1};$$

and solving for η from this equation, we obtain $\eta = -\nu^2$; hence, we have

$$C = (C + D)\frac{\nu^2}{\nu^2 - 1} = \frac{c}{Lab}\frac{\nu^3}{1-\nu^2}, \quad D = \frac{C+D}{1-\nu^2} = \frac{-c}{Lab}\frac{1}{1-\nu^2},$$

and

$$j_n = \frac{1}{\beta_n} = \frac{1}{C\nu^n + D\nu^{-n}} = -\frac{Lab}{c}\frac{(1-\nu^2)\nu^n}{1-\nu^{2n+2}}. \qquad (47)$$

As the potential due to the charges i_0, j_0, &c., is the same on each of the surfaces (A) and (B) as that due to the actual distributions on those surfaces, the total mass E_a on (A) must be equal to the sum of the charges at the interior points A, I_1, I_2, &c., and the total mass E_b on (B) to the sum of the charges at J_0, J_1, &c. Hence

$$E_a = \Sigma i_n = La(1-\lambda^2)\,\Sigma\frac{\nu^n}{1-\lambda^2\nu^{2n}},$$

$$E_b = \Sigma j_n = -\frac{Lab}{c}(1-\nu^2)\,\Sigma\frac{\nu^n}{1-\nu^{2n+2}}.$$

If we put

$$q_{11} = a(1-\lambda^2)\sum_0^\infty\frac{\nu^n}{1-\lambda^2\nu^{2n}}, \quad q_{12} = -\frac{ab}{c}(1-\nu^2)\sum_0^\infty\frac{\nu^n}{1-\nu^{2n+2}}, \qquad (48)$$

we see that q_{12} is symmetrical with respect to a and b, and we have

$$E_a = q_{11}L. \quad E_b = q_{12}L. \qquad (49)$$

If the sphere (B) were at potential M, and (A) at potential

zero, we should find in like manner for the total charges E'_a and E'_b on (A) and (B), the equations $E'_a = q_{12}M$, $E'_b = q_{22}M$, where

$$q_{22} = b(1-\mu^2)\sum_0^\infty \frac{\nu^n}{1-\mu^2\nu^{2n}}, \quad \mu = \frac{b+\nu a}{c}. \qquad (50)$$

Since for each of the distributions considered above the potential is constant on each sphere, we may suppose the two distributions to be superposed, and thus for E_1 and E_2, the total charges on the spheres when insulated and at potentials L and M, we have the equations

$$E_1 = q_{11}L + q_{12}M, \quad E_2 = q_{12}L + q_{22}M. \qquad (51)$$

The expressions for q_{11}, q_{12}, and q_{22}, given in (48) and (50), may, as Mr. F. Purser has shown, be reduced to more convenient forms in the following manner:

If we put $4c^2k^2 = a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + c^2a^2)$, we have

$$\nu = \frac{c^2 - a^2 - b^2 - 2ck}{2ab}, \quad \nu^{-1} = \frac{c^2 - a^2 - b^2 + 2ck}{2ab};$$

whence $\dfrac{1-\nu^2}{\nu} = \dfrac{2ck}{ab}$, and since $\lambda = \dfrac{a+\nu b}{c} = \dfrac{\nu c}{\nu a + b}$,

we have

$$\lambda^2 = \frac{\nu a + \nu^2 b}{\nu a + b}, \quad \text{and} \quad 1 - \lambda^2 = \frac{b(1-\nu^2)}{\nu a + b} = \frac{2ck\nu}{a(\nu a + b)} = \frac{2k}{a}\lambda.$$

In like manner we get

$$1 - \mu^2 = \frac{2k}{b}\mu.$$

Again

$$\frac{\lambda\nu^n}{1-\lambda^2\nu^{2n}} = \lambda\nu^n(1 + \lambda^2\nu^{2n} + \lambda^4\nu^{4n} + \&c.) = \Sigma_0^\infty \lambda^{2m+1}\nu^{n(2m+1)},$$

whence

$$\sum_1^\infty \frac{\lambda\nu^n}{1-\lambda^2\nu^{2n}} = \sum_0^\infty \sum_1^\infty \lambda^{2m+1}\nu^{n(2m+1)} = \sum_0^\infty \frac{(\lambda\nu)^{2n+1}}{1-\nu^{2n+1}}.$$

In like manner, we obtain

$$\sum_0^\infty \frac{\nu^{n+1}}{1-\nu^{2n+2}} = \sum_0^\infty \frac{\nu^{2n+1}}{1-\nu^{2n+1}}.$$

Substituting in (48) and (50) for $1-\lambda^2$, $1-\nu^2$, $1-\mu^2$, and the infinite series, in accordance with the equations obtained above, we get

$$\left.\begin{aligned} q_{11} &= a + 2k\sum_0^\infty \frac{(\lambda\nu)^{2n+1}}{1-\nu^{2n+1}} \\ q_{12} &= -\ 2k\ \sum_0^\infty \frac{\nu^{2n+1}}{1-\nu^{2n+1}} \\ q_{22} &= b + 2k\sum_0^\infty \frac{(\mu\nu)^{2n+1}}{1-\nu^{2n+1}} \end{aligned}\right\}. \tag{52}$$

Other methods of finding the values of q_{11}, &c., will be found in the Examples.

Examples.

1. Obtain directly, in terms of a, b, and c, the first three terms in q_{11}, q_{12}, and q_{22}.

Since $q_{11}L = i_0 + i_1 + i_2 +$ &c., and $q_{12}L = j_0 + j_1 + j_2 +$ &c., we have to find i_0, j_0, &c. We have then

$$i_0 = La, \qquad BI_0 = BA = c, \qquad j_0 = -\frac{b}{c}\,i_0 = -\frac{Lab}{c},$$

$$BJ_0 = \frac{b^2}{c}, \quad AJ_0 = c - BJ_0 = \frac{c^2-b^2}{c}, \quad i_1 = -\frac{a}{AJ_0}\,j_0 = \frac{La^2b}{c^2-b^2},$$

$$AI_1 = \frac{a^2}{AJ_0} = \frac{a^2c}{c^2-b^2}, \quad BI_1 = c - AI_1 = \frac{c(c^2-a^2-b^2)}{c^2-b^2},$$

$$j_1 = -\frac{b}{BI_1}\,i_1 = -\frac{La^2b^2}{c(c^2-a^2-b^2)}, \qquad BJ_1 = \frac{b^2}{BI_1} = \frac{b^2(c^2-b^2)}{c(c^2-a^2-b^2)},$$

$$AJ_1 = c - BJ_1 = \frac{(c^2-b^2)^2-a^2c^2}{c(c^2-a^2-b^2)}, \quad i_2 = -\frac{a}{AJ_1}\,j_1 = \frac{La^3b^2}{(c^2-b^2+ac)(c^2-b^2-ac)},$$

$$AI_2 = \frac{a^2}{AJ_1} = \frac{a^2c(c^2-a^2-b^2)}{(c^2-b^2)^2-a^2c^2},$$

$$BI_2 = c - AI_2 = \frac{c\{c^4+a^4+b^4-2a^2c^2-2b^2c^2+a^2b^2\}}{(c^2-b^2)^2-a^2c^2},$$

$$j_2 = -\frac{b}{BI_2}\,i_2 = -\frac{La^3b^3}{c(c^2-a^2-b^2+ab)(c^2-a^2-b^2-ab)}.$$

Hence we obtain

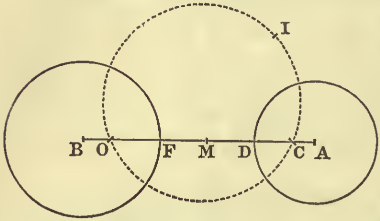
$$q_{11} = a + \frac{a^2b}{c^2-b^2} + \frac{a^3b^2}{(c^2-b^2+ac)(c^2-b^2-ac)},$$

$$q_{12} = -\frac{ab}{c} - \frac{a^2b^2}{c(c^2-a^2-b^2)} - \frac{a^3b^3}{c(c^2-a^2-b^2+ab)(c^2-a^2-b^2-ab)};$$

and interchanging a and b in the expression for q_{11}, we have

$$q_{22} = b + \frac{ab^2}{c^2 - a^2} + \frac{a^2 b^3}{(c^2 - a^2 + bc)(c^2 - a^2 - bc)}$$

2. Two spheres (A) and (B) are at potential zero under the influence of a charge E situated at a point I outside them; find the total charge on each sphere.



Let A and B denote the centres of the spheres, a and b their radii, c the distance AB, M the point in AB from which tangents to the two spheres are equal, O and C points on AB such that $MO = MC =$ tangent to either sphere from M, D and F the points in which AB meets the spheres, and let O be inside (B); then $b^2 = BM^2 - OM^2 = BO \,.\, BC$, and therefore O and C are images in (B); also, as can be shown in like manner, they are images in (A).

Invert the system from O, and we obtain two concentric spheres (A') and (B') whose centre is C' the point inverse to C, and which are at potential zero under the influence of a charge E' at the point I' inverse to I. The space outside (A) and (B) corresponds to the space outside (A') and inside (B') in the inverse system. Hence (A') and (B') are at potential zero under the influence of a charge E', at a point I', situated between them, and therefore if the images of I' be denoted by I'_1, J'_1, their images by J'_2, I'_2, &c., and the corresponding charges by i'_1, &c., by Art. 110, we have

$$i'_n = \pm \sqrt{\left(\frac{C'I'_n}{C'I'}\right)}\, E', \qquad j'_n = \pm \sqrt{\left(\frac{C'J'_n}{C'I'}\right)}\, E',$$

but, by similar triangles,

$$\frac{CI_n}{OI_n} = \frac{C'I'_n}{OC'}, \tag{a}$$

and also $\dfrac{CI}{OI} = \dfrac{C'I'}{OC'}$; whence dividing the members of one of these equations by those of the other, we have

$$\frac{C'I'_n}{C'I'} = \frac{CI_n}{OI_n}\,\frac{OI}{CI}.$$

Again

$$i_n = \frac{OI_n}{R}\, i'_n, \quad \text{and} \quad E' = \frac{R}{OI} E.$$

Hence we obtain

$$i_n = \pm\, E \sqrt{\left(\frac{OI_n \,.\, CI_n}{OI \,.\, CI}\right)}. \tag{b}$$

The value of the product $CI_n \,.\, OI_n$ can be obtained in the following manner: Since the points I', I'_1, J'_1, &c., lie on a straight line passing through C', the points I, I_1, &c., lie on a circle passing through O and C. Let ϕ denote the

angle of the segment of this circle standing on OC, a' and b' the radii of the spheres (A') and (B'), and f' the distance $C'I'$; also let

$$k = MO, \qquad \alpha = \log \frac{CD}{OD}, \qquad \beta = \log \frac{CF}{OF}, \qquad \theta = \log \frac{CI}{OI},$$

$$\varpi = \log \frac{1}{\mu'} = \log \frac{b'}{a'}, \qquad \epsilon = \log \frac{f'}{a'}, \qquad \theta_n = \log \frac{CI_n}{OI_n}, \quad \psi_n = \log \frac{CJ_n}{OJ_n};$$

then, by similar triangles and division as above,

$$\frac{b'}{a'} = \frac{C'F'}{C'D'} = \frac{CF}{CD}\frac{OD}{OF},$$

and

$$\frac{f'}{a'} = \frac{C'I'}{C'D'} = \frac{CI}{CD}\frac{OD}{OI};$$

whence

$$\varpi = \log \frac{CF}{OF} - \log \frac{CD}{OD} = \beta - \alpha,$$

$$\epsilon = \log \frac{CI}{OI} - \log \frac{CD}{OD} = \theta - \alpha.$$

Again, by (a)

$$\frac{CI_n}{OI_n} = \frac{C'I'_n}{OC'} = \frac{C'I'_n}{C'I'}\frac{C'I'}{OC'} = \frac{C'I'_n}{C'I'}\frac{CI}{OI};$$

but, by Art. 125,

$$\frac{C'I'_{2n}}{C'I'} = \mu'^{2n}, \qquad \frac{C'I'_{2n-1}}{C'I'} = \mu'^{2(n-1)}\frac{a'^2}{f'^2};$$

whence $\theta_{2n} = \theta - 2n\varpi$, $\theta_{2n-1} = \theta - 2\epsilon - 2(n-1)\varpi = 2\alpha - \theta - 2(n-1)\varpi$.

In like manner,

$$\psi_{2n} = \theta + 2n\varpi, \quad \psi_{2n-1} = \theta + 2n\varpi - 2\epsilon = 2\alpha - \theta + 2n\varpi = 2\beta - \theta + 2(n-1)\varpi.$$

From the definition of θ_n, we have

$$\frac{CI_n}{OI_n} = e^{\theta_n}; \text{ whence } \frac{OI_n}{CI_n} = e^{-\theta_n}, \text{ and } \frac{CI_n^2 + OI_n^2}{2CI_n \,.\, OI_n} = \cosh\theta_n;$$

also, from the triangle CI_nO, we get $2CI_n \,.\, OI_n \cos\phi = CI_n^2 + OI_n^2 - 4k^2$; eliminating $CI_n^2 + OI_n^2$ from these equations, we obtain

$$CI_n \,.\, OI_n = \frac{2k^2}{\cosh\theta_n - \cos\phi}; \text{ hence, we have } i_n = \pm E\left(\frac{\cosh\theta - \cos\phi}{\cosh\theta_n - \cos\phi}\right)^{\frac{1}{2}},$$

and in a similar manner we get

$$j_n = \pm E\left(\frac{\cosh\theta - \cos\phi}{\cosh\psi_n - \cos\phi}\right)^{\frac{1}{2}}$$

Since the points I_1 I_2, &c., are all inside the sphere (A), and the points J_1, J_2, &c., inside (B), if E_a and E_b denote the total charges on (A) and (B), we have $E_a = \Sigma i_n$, $E_b = \Sigma j_n$; whence

$$E_a = E(\cosh\theta - \cos\phi)^{\frac{1}{2}}$$

$$\sum_1^\infty (\{\cosh(\theta - 2n\varpi) - \cos\phi\}^{-\frac{1}{2}} - \{\cosh[2\alpha - \theta - 2(n-1)\varpi] - \cos\phi\}^{-\frac{1}{2}}),$$

$$E_b = E(\cosh\theta - \cos\phi)^{\frac{1}{2}}$$

$$\sum_1^\infty (\{\cosh(\theta + 2n\varpi) - \cos\phi\}^{-\frac{1}{2}} - \{\cosh[2\beta - \theta + 2(n-1)\varpi] - \cos\phi\}^{-\frac{1}{2}}).$$

3. Apply the method of the preceding example to determine the total charges on two insulated spheres, one of which (A) is at potential L, and the other (B) at potential zero.

The potential at the surfaces of (A) and (B) is the same as that due to a charge La at A together with its successive images in (B) and (A). The mathematical relations between the positions and charges of these images and those of the inverse system which have been investigated in the last example still hold good, though here the point which is the inverse of A is inside both (A') and (B'). In this case all the images J_1, I_2, &c. are on the straight line AB, in the series I_n every suffix is even, and in the series J_n every suffix is odd; also every J lies between B and O, and every I between C and A; hence

$$OI_n - CI_n = CJ_n - OJ_n = OC = 2k, \text{ and } \cos\phi = 1.$$

Since $\qquad MA^2 - MB^2 = a^2 - b^2, \quad \text{and} \quad MA + MB = c,$

we get
$$MA = \frac{c^2 + a^2 - b^2}{2c}$$

and
$$k^2 = MO^2 = MA^2 - a^2 = \frac{a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2}{4c^2}.$$

Again, in this case, $\theta = \log\dfrac{CA}{OA}$, but $\dfrac{CA}{DA} = \dfrac{DA}{OA}$, whence $\dfrac{CA}{OA} = \left(\dfrac{DA}{OA}\right)^2$,

and
$$\frac{DA - CA}{DA} = \frac{OA - DA}{OA}, \quad \text{that is,} \quad \frac{DC}{OD} = \frac{DA}{OA};$$

therefore $\theta = 2\log\dfrac{DC}{OD} = 2\alpha$. Again,

$$e^\alpha = \frac{DC}{OD} = \frac{DA}{OA} = \frac{a}{k + \sqrt{k^2 + a^2}} = \frac{\sqrt{k^2 + a^2} - k}{a}, \text{ and } e^{-\alpha} = \frac{\sqrt{k^2 + a^2} + k}{a},$$

whence $\quad \sinh\alpha = -\dfrac{k}{a}$.

Since $\theta = 2\alpha$, we have

$$\cosh\theta - 1 = 2\sinh^2\alpha, \quad \cosh(\theta - 2n\varpi) - 1 = 2\sinh^2(\alpha - n\varpi),$$

$$\cosh(2\alpha - \theta + 2n\varpi) - 1 = \cosh 2n\varpi - 1 = 2\sinh^2 n\varpi;$$

and as in this case $E = La$, we have

$$E(\cosh\theta - \cos\phi)^{\frac{1}{2}} = Lk\sqrt{2}.$$

Hence, for E_a and E_b the charges on the spheres (A) and (B), we get the equations

$$E_a = La + Lk \sum_1^\infty \frac{1}{\sinh(n\varpi - a)} = Lk \sum_0^\infty \frac{1}{\sinh(n\varpi - a)},$$

$$E_b = - Lk \sum_1^\infty \frac{1}{\sinh n\varpi}.$$

4. An insulated conductor formed of two spheres in contact is charged to potential L; express the charges on the two spheres by means of Eulerian Integrals.

Investigations of most of the properties of Eulerian Integrals are given in Williamson's Integral Calculus, Chapter VI. Some of those required in the present case are not to be found in that treatise, and of these a brief exposition is here supplied for the convenience of the student. For fuller information the reader is referred to Williamson's Article on the Integral Calculus in the *Encyclopædia Britannica.*

The second Eulerian Integral $\Gamma(x)$ being defined by the equation

$$\Gamma(x) = \int_0^\infty \varepsilon^{-\theta}\theta^{x-1}\,d\theta;$$

if we assume $\varepsilon^{-\theta} = z$, we may write

$$\Gamma(x) = \int_0^1 \left(\log\frac{1}{z}\right)^{x-1} dz;\quad \text{now } \log\frac{1}{z} = m\left(1 - z^{\frac{1}{m}}\right)$$

when $m = \infty$, for putting $m = \frac{1}{\xi}$, we have $m\left(1 - z^{\frac{1}{m}}\right) = \frac{1 - z^\xi}{\xi}$, the value of which when $\xi = 0$ is $-\log z$.

Hence $\Gamma(x) = \int_0^1 m^{x-1}\left(1 - z^{\frac{1}{m}}\right)^{x-1} dz$ when $m = \infty$.

If we assume $z = y^m$, the integral

$$\int_0^1 m^{x-1}\left(1 - z^{\frac{1}{m}}\right)^{x-1} dz \quad \text{becomes} \quad m^x\int_0^1 y^{m-1}(1-y)^{x-1}\,dy;$$

integrating by parts, we get

$$\int_0^1 y^{m-1}(1-y)^{x-1}\,dy = -\Big|_0^1 \frac{y^{m-1}(1-y)^x}{m+x-1} + \frac{m-1}{m+x-1}\int_0^1 y^{m-2}(1-y)^{x-1}\,dy,$$

and, as the part outside the integral sign vanishes at both limits, by successive applications of this process, when m is an integer, we obtain for the right hand member of the above equation the value

$$\frac{(m-1)(m-2)\,.\,.\,.\,.\,.\,1}{(m+x-1)(m+x-2)\,.\,.\,.\,.\,.\,(x+1)}\int_0^1 (1-y)^{x-1}\,dy.$$

Hence, $$\Gamma(x) = m^x\frac{1\,.\,2\,.\,3\,.\,.\,.\,.\,(m-1)}{x(x+1)(x+2)\,.\,.\,.\,.\,(x+m-1)}, \qquad (a)$$

and $$\frac{d}{dx}\log\Gamma(x) = \log m - \frac{1}{x} - \frac{1}{x+1}\,.\,.\,.\,.\, - \frac{1}{x+m-1}, \qquad (b)$$

where m is an infinite integer.

Changing x into $1+x$ in (b), expanding in powers of x, and putting

$S_1 = 1 + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{m}$, $S_2 = 1 + (\frac{1}{2})^2 + (\frac{1}{3})^2 + \text{\&c.}$, $S_3 = 1 + (\frac{1}{2})^3 + \text{\&c.}$, &c. = &c.,

we have

$$\frac{d}{dx} \log \Gamma(x+1) = \log m - S_1 + S_2 x - S_3 x^2 + \text{\&c.} \tag{c}$$

If we suppose x less than unity $\frac{d}{dx} \log \Gamma(x+1)$ is finite, and the series in x convergent, and therefore when m is an infinite integer $\log m - S_1$ is a finite constant whose value may be denoted by $-\gamma$; accordingly, returning to the original series, we have

$$\frac{d}{dx} \log \Gamma(x) = -\gamma + \sum_1^\infty \frac{1}{n} - \sum_0^\infty \frac{1}{x+n}, \tag{d}$$

$$\frac{d}{dx} \log \Gamma(x+1) = -\gamma + \sum_1^\infty \frac{1}{n} - \sum_1^\infty \frac{1}{x+n}. \tag{e}$$

If we integrate (c) we get

$$\log \Gamma(x+1) = -\gamma x + \tfrac{1}{2} S_2 x^2 - \tfrac{1}{3} S_3 x^3 + \text{\&c.}; \tag{f}$$

whence

$$\log \Gamma(1-x) = \gamma x + \tfrac{1}{2} S_2 x^2 + \tfrac{1}{3} S_3 x^3 + \text{\&c.},$$

and therefore

$$\log \Gamma(1+x)\, \Gamma(1-x) = 2 \left\{ S_2 \frac{x^2}{2} + S_4 \frac{x^4}{4} + \text{\&c.} \right\}. \tag{g}$$

Since $$\sin \pi x = \pi x (1-x^2) \left(1 - \frac{x^2}{2^2}\right) \left(1 - \frac{x^2}{3^2}\right), \text{\&c.},$$

it follows from (g) that

$$\Gamma(1+x)\, \Gamma(1-x) = \frac{\pi x}{\sin \pi x};$$

also, substituting in (f), we have

$$\log \Gamma(x+1) = -\gamma x + \tfrac{1}{2} \log \frac{\pi x}{\sin \pi x} - S_3 \frac{x^3}{3} - S_5 \frac{x^5}{5} - \text{\&c.} \tag{h}$$

Again, from the definition of $\Gamma(x)$ by integration by parts we have

$$\Gamma(x+1) = x\, \Gamma(x),$$

and therefore

$$\Gamma(x)\, \Gamma(1-x) = \frac{1}{x}\, \Gamma(1+x)\, \Gamma(1-x) = \frac{\pi}{\sin \pi x};$$

whence, making $x = \frac{1}{2}$, we have

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \quad \text{and also} \quad \Gamma(1+\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi}.$$

Substituting in (h) and reducing, we obtain

$$\gamma = \log 2 - 2\{\tfrac{1}{3}(\tfrac{1}{2})^3 S_3 + \tfrac{1}{5}(\tfrac{1}{2})^5 S_5 + \&c.\} = 0{\cdot}57712.$$

By (27), Art. 124,

$$E_a = \frac{Lab}{a+b}\sum_1^\infty \left(\frac{1}{n+\mu-1} - \frac{1}{n}\right) = La + \frac{Lab}{a+b}\sum_1^\infty \left(\frac{1}{\mu+n} - \frac{1}{n}\right),$$

since $\mu = \dfrac{b}{a+b}$. Hence by (d) and (e) we have

$$E_a = -\frac{Lab}{a+b}\left\{\gamma + \frac{d}{d\mu}\log\,\Gamma(\mu)\right\} = La - \frac{Lab}{a+b}\left\{\gamma + \frac{d}{d\mu}\log\,\Gamma(1+\mu)\right\}.$$

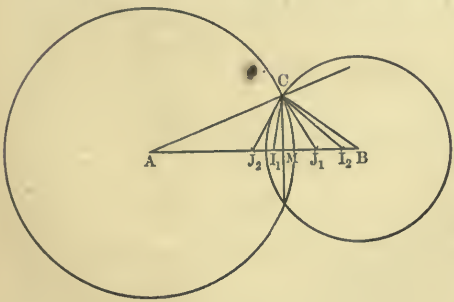
Again, $$E_b = \frac{Lab}{a+b}\sum_1^\infty \left(\frac{1}{n-\mu} - \frac{1}{n}\right) = -\frac{Lab}{a+b}\left\{\gamma - \frac{d}{d\mu}\log\Gamma(1-\mu)\right\};$$

and therefore

$$E_a - E_b = -\frac{Lab}{a+b}\frac{d}{d\mu}\log\{\Gamma(\mu)\,\Gamma(1-\mu)\} = \frac{Lab}{a+b}\frac{d}{d\mu}\log\left(\frac{\sin\mu\pi}{\pi}\right)$$

$$= \pi L\,\frac{ab}{a+b}\cot\frac{\pi b}{a+b}$$

5. If two spheres (A) and (B) intersect, and J_1 be the point which is the image in (B) of A the centre of (A), I_1 the image of J_1 in (A), J_2 the image of I_1 in B, and so on, and if a charge e be placed at A, and charges, which are the successive images of e in (B) and (A), at J_1, I_1, J_2, I_2, &c., find expressions for the sum E_a of the charges at A, I_1, I_2, &c., and the sum E_b of the charges at J_1, J_2, &c.



If we denote the charges at I_1, J_1, &c. by i_1, j_1, &c.,

we have $$\frac{e}{CA} = \frac{-j_1}{CJ_1} = \frac{i_1}{CI_1} = \frac{-j_2}{CJ_2} = \&c.,$$

where C is a point common to the two spheres.

Again, if J_n, I_n, J_{n+1}, and I_{n+1}, be successive images, it is easy to see that

$$BJ_{n+1}C = BCI_n = BCA - ACI_n = BCA - AJ_nC,$$

and that

$$AI_{n+1}C = ACJ_{n+1} = BCA - BCJ_{n+1} = BCA - BI_nC,$$

when the radii of the spheres and the distances between their centres are such that the successive images lie between A and B, and that the angle BCA is obtuse.

If A and C denote the angles BAC and the supplement of BCA, we have, then

$$AJ_1C = C, \quad BJ_2C = \pi - 2C, \quad AJ_2C = 2C, \quad BJ_3C = \pi - 3C, \text{ \&c.},$$

$$BAC = A, \quad AI_1C = \pi - C - A, \quad BI_1C = A + C, \quad AI_2C = \pi - A - 2C, \text{ \&c.}$$

If p denote the perpendicular CM on AB, we have

$$CA = \frac{p}{\sin A}, \quad CJ_1 = \frac{p}{\sin AJ_1C}, \quad CI_1 = \frac{p}{\sin BI_1C}, \text{ \&c.};$$

whence

$$E_a = \frac{ep}{a}\left(\frac{1}{\sin A} + \frac{1}{\sin (A+C)} + \text{\&c.}\right) = \frac{ep}{a}\sum_0^\infty \frac{1}{\sin (A + nC)},$$

$$E_b = \frac{-ep}{a}\sum_1^\infty \frac{1}{\sin nC}.$$

These expressions are due to Mr. F. Purser.

6. From the results obtained in the last Example deduce expressions for the total charges on two insulated spheres, one of which (A) is at potential L, and the other (B) at potential zero.

When the spheres (A) and (B) do not intersect the point C, the perpendicular p, and the angles A, B, and C become imaginary, and if $i = \sqrt{-1}$, we may put $A = i\alpha$, $C = i\gamma$, where α and γ are real, then

$$\cosh \alpha = \cos A = \frac{a^2 + c^2 - b^2}{2ac}, \quad i \sinh \alpha = \sin A = \left\{\frac{2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4}{4a^2c^2}\right\}^{\frac{1}{2}},$$

$$\cosh \gamma = \cos C = \frac{c^2 - a^2 - b^2}{2ab}, \quad i \sinh \gamma = \sin C = \left\{\frac{2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4}{4a^2b^2}\right\}^{\frac{1}{2}}.$$

Hence, if

$$k^2 = \frac{a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + c^2a^2)}{4c^2},$$

we have

$$\sinh \alpha = \frac{k}{a}, \qquad \sinh \gamma = \frac{kc}{ab},$$

$$\sin (A + nC) = \sinh i(\alpha + n\gamma) = i \sinh (\alpha + n\gamma), \quad \sinh nC = \sinh in\gamma = i \sinh n\gamma;$$

also,

$$p = \frac{ab}{c}\sin C = \frac{iab}{c}\sinh \gamma = ik.$$

Hence, putting $e = La$, and substituting in the expressions for E_a and E_b given in the last Example, we get

$$E_a = Lk\sum_0^\infty \frac{1}{\sinh (\alpha + n\gamma)}, \quad E_b = -Lk\sum_1^\infty \frac{1}{\sinh n\gamma}.$$

This agrees with Ex. 3, where α has the same meaning as α in the present Example with its sign changed, and $\varpi = \gamma$.

The method adopted here is due to Mr. F. Purser.

7. If $\tan \epsilon = \frac{ab}{kc}$, $\tan \theta_0 = \frac{a}{k}$, $\tan \frac{1}{2}\theta_{n+1} = \tan \frac{1}{2}\epsilon \tan \frac{1}{2}\theta_n$, $\tan \psi_1 = \tan \epsilon$, $\tan \frac{1}{2}\psi_{n+1} = \tan \frac{1}{2}\epsilon \tan \frac{1}{2}\psi_n$, show that E_a and E_b in the preceding Example may be expressed by the equations

$$E_a = Lk \sum_0^\infty \tan \theta_n, \quad E_b = -Lk \sum_1^\infty \tan \psi_n.$$

If ξ_n and η_n denote the distances of I_n and J_n from M, the point of intersection of AB with the radical plane of the spheres (A) and (B), it is plain that when the spheres intersect

$$CI_n = (\xi_n^2 + p^2)^{\frac{1}{2}}, \qquad CJ_n = (\eta_n^2 + p^2)^{\frac{1}{2}}.$$

Hence, in Ex. 5, we have

$$i_n = \frac{e}{a} (\xi_n^2 + p^2)^{\frac{1}{2}}, \qquad j_n = \frac{-e}{a} (\eta_n^2 + p^2)^{\frac{1}{2}}.$$

When the spheres (A) and (B) do not intersect, as in Ex. 6, we have

$$p = k\sqrt{-1}, \quad \text{and} \quad i_n = L(\xi_n^2 - k^2)^{\frac{1}{2}}, \quad j_n = -L(\eta_n^2 - k^2)^{\frac{1}{2}}. \qquad (a)$$

Now $\xi_n = MA - I_nA$ (see figure of Ex. 2) and, as in Art. 126, putting $I_nA = f_n$, $J_nB = h_n$, and putting also $\xi_0 = MA$, $\eta_0 = MB$, as in Ex. 3, we have

$$\xi_0 = \frac{c^2 + a^2 - b^2}{2c}, \quad \eta_0 = \frac{c^2 + b^2 - a^2}{2c};$$

and by (38), Art. 126, we get

$$f_{n+1} = \frac{a^2(c - f_n)}{c^2 - b^2 - cf_n}.$$

Hence, we obtain

$$\xi_{n+1} = \xi_0 - f_{n+1} = \xi_0 - \frac{a^2(c - \xi_0 + \xi_n)}{c^2 - b^2 - c(\xi_0 - \xi_n)} = \frac{k^2c + \frac{1}{2}(c^2 - a^2 - b^2)\xi_n}{\frac{1}{2}(c^2 - a^2 - b^2) + c\xi_n};$$

and if we put $\xi_n = k \sec \theta_n$, we get

$$\cos \theta_{n+1} = \frac{2ck + (c^2 - a^2 - b^2)\cos \theta_n}{c^2 - a^2 - b^2 + 2ck \cos \theta_n} = \frac{\cos \epsilon + \cos \theta_n}{1 + \cos \epsilon \cos \theta_n},$$

and therefore $\tan \frac{1}{2}\theta_{n+1} = \tan \frac{1}{2}\epsilon \tan \frac{1}{2}\theta_n$.

If we put $\eta_n = k \sec \psi_n$, in like manner we obtain $\tan \frac{1}{2}\psi_{n+1} = \tan \frac{1}{2}\epsilon \tan \frac{1}{2}\psi_n$.
From equations (a) we have $\qquad i_n = Lk \tan \theta_n, \quad j_n = -Lk \tan \psi_n$;

and, as $i_0 = La$, $j_1 = -\frac{Lab}{c}$,

we get $$\tan \theta_0 = \frac{a}{k}, \quad \tan \psi_1 = \frac{ab}{kc} = \tan \epsilon.$$

The expressions for E_a and E_b given in this Example are due to Mr. F. Purser.

8. Find, by means of the last Example, the expressions for q_{11}, q_{12}, and q_{22} given in Art. 126.

If we put

$$\lambda = \tan \tfrac{1}{2}\theta_0, \quad \nu = \tan \tfrac{1}{2}\epsilon, \quad \mu = \tan \tfrac{1}{2}\phi_0, \quad \text{where} \quad \tan \phi_0 = \frac{b}{k},$$

remembering that

$$4c^2k^2 = a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + c^2a^2),$$

we get

$$\lambda = \frac{\sqrt{(a^2 + k^2)} - k}{a} = \frac{c^2 + a^2 - b^2 - 2ck}{2ac},$$

$$\nu = \frac{\sqrt{(c^2k^2 + a^2b^2)} - ck}{ab} = \frac{c^2 - a^2 - b^2 - 2ck}{2ab}.$$

Hence ν has here the same meaning as in equation (40); also, λ in Art. 126, is defined by the equation $\lambda = \dfrac{a + \nu b}{c}$, and therefore has the same meaning as in the present Example. A similar result holds good for μ. By Ex. 7, we have, then,

$$q_{11} = a + k \sum\nolimits_1^\infty \tan \theta_n = a + 2k\sum\nolimits_1^\infty \frac{\tan \frac{1}{2}\theta_n}{1 - \tan^2 \frac{1}{2}\theta_n} = a + 2k \sum\nolimits_1^\infty \frac{\lambda\nu^n}{1 - \lambda^2\nu^{2n}},$$

$$-q_{12} = k\sum\nolimits_1^\infty \tan \psi_n = 2k \sum\nolimits_1^\infty \frac{\tan \frac{1}{2}\psi_n}{1 - \tan^2 \frac{1}{2}\psi_n} = 2k \sum\nolimits_1^\infty \frac{\nu^n}{1 - \nu^{2n}},$$

$$q_{22} = b + 2k \sum\nolimits_1^\infty \frac{\mu\nu^n}{1 - \mu^2\nu^{2n}};$$

and hence, as in Art. 126, we obtain equations (52).

9. A circular disk is at potential zero under the influence of a charge e situated at a point O in the perpendicular to the plane of the disk through its centre; find the distribution of electricity on the disk.

Let a denote the radius of the disk, and h and R the distances of its centre, and a point on its edge from O. If we invert from O, taking R as the radius of inversion, the disk is inverted into a bowl whose base is the disk. If the bowl be at constant potential L', by Ex. 3, Art. 123, the densities of the distribution on its outer and inner surfaces are given at any point P', by the equations

$$\sigma'_1 = \frac{L'}{2\pi f}\left\{1 + \frac{1}{\pi}\left[\sqrt{\left(\frac{\zeta_0}{\zeta_{P'}}\right)} - \tan^{-1}\sqrt{\left(\frac{\zeta_0}{\zeta_{P'}}\right)}\right]\right\},$$

$$\sigma'_2 = \sigma'_1 - \frac{L'}{2\pi f}.$$

It is easy to see that

$$\frac{\zeta_0}{\zeta_{P'}} = \frac{r}{r' - r} = \frac{r^2}{R^2 - r^2};$$

whence we obtain

$$\tan^{-1}\sqrt{\left(\frac{\zeta_0}{\zeta_{P'}}\right)} = \sin^{-1}\frac{r}{R};$$

also, we have $f = \dfrac{R^2}{h}$, and $R^2 = a^2 + h^2$. By (*b*), 2°, Art. 119, if $L' = -\dfrac{e}{R}$, the disk is at potential zero, and σ_1, the density of the distribution on the side which is next O, is given at any point P by the equation

$$\sigma_1 = \left(\frac{R}{r}\right)^3 \sigma_1' = \frac{-eh}{2\pi r^3}\left\{1 + \frac{1}{\pi}\left[\frac{r}{\sqrt{(a^2 - \varpi^2)}} - \sin^{-1}\frac{r}{\sqrt{(a^2 + h^2)}}\right]\right\},$$

where r and ϖ are the distances of P from O and the centre of the disk. The density σ_2 of the distribution on the side remote from O is connected with σ_1 by the equation

$$\sigma_2 = \sigma_1 + \frac{eh}{2\pi r^3}.$$

10. If an infinitely thin conductor, on which mass is distributed, be at potential zero under the influence of mass not on the conductor, show that the densities σ_1 and σ_2 of the distributions on each side of the conductor, and the total density σ, at the same point P, are connected with the potential V of the entire system of mass wherever situated by the equations

$$4\pi\sigma_1 + \frac{dV}{d\nu_1} = 0, \quad 4\pi\sigma_2 + \frac{dV}{d\nu_2} = 0, \quad \sigma_1 + \sigma_2 = \sigma,$$

where ν_1 and ν_2 are normals drawn from the surface on each side at the point P.

CHAPTER VII.

SYSTEMS OF CONDUCTORS.

127. **Distribution on Charged Conductors.**—If a system of charged insulated conductors be in a state of electric equilibrium, the potential is constant on the surface of each conductor, and there is only one possible distribution of electricity which produces a potential having given values at these surfaces, since (Art. 64) there is only one possible potential in external space, and its differential coefficient determines the density of the distribution at any point on a conductor. Again, if the total charge on each conductor be given, there is only one possible distribution of electricity consistent with equilibrium. This proposition is a generalization of that given in Art. 75, and is proved in a similar manner.

In fact, if S_1, S_2, &c. be the surfaces of the conductors, e_1, e_2, &c., the given charges, V and V' the potentials due to two supposed distributions of these charges consistent with equilibrium, we have $V = C_1$, $V' = C'_1$, at S_1; $V = C_2$, $V' = C'_2$, at S_2, &c.; and ν being the normal to a surface drawn into external space,

$$\int \frac{dV}{d\nu}\, dS_1 = -4\pi e_1 = \int \frac{dV'}{d\nu}\, dS_1, \text{ \&c.};$$

whence

$$\int (V - V')\, \frac{d}{d\nu}\, (V - V')\, dS_1 = 0, \text{ \&c.}$$

Again, throughout the whole of space $\mathfrak{S}$ outside the conductors $\nabla^2 V = \nabla^2 V' = 0$; hence, if $\phi = V - V'$, we have

$$\int \phi\, \frac{d\phi}{d\nu}\, dS_1 + \int \phi\, \frac{d\phi}{d\nu}\, dS_2 + \text{\&c.} + \int \phi \nabla^2 \phi\, d\mathfrak{S} = 0;$$

and therefore, by (9), Art. 58, ϕ is constant throughout the

whole of external space, and $\frac{dV}{d\nu} = \frac{dV'}{d\nu}$ at every point of each conductor; whence the two supposed distributions must be identical.

128. **Relation between Charges and Potentials.**—If V_1, V_2, &c., be the values of the potential on the conductors A_1, A_2, &c., these values are connected with the charges e_1, e_2, &c., by linear equations whose coefficients form a symmetric determinant.

To prove this, suppose the unit charge of electricity to be imparted to the conductor A_1, the others remaining uncharged, then, by Art. 127, the potentials $V_1, V_2, \ldots V_n$ take definite values which may be denoted by $p_{11}, p_{21}, \ldots p_{n1}$. If now the density of the distribution at each point on each conductor be altered in the ratio $e_1 : 1$, the potentials are altered in the same ratio; and we have

$$V_1 = p_{11}e_1, \quad V_2 = p_{21}e_1, \ldots V_n = p_{n1}e_1,$$

the charge on A_1 being now e_1, and that on each of the other conductors zero as before. In a similar manner it appears that, if there be a charge e_2 on A_2, the remaining conductors being uncharged, the potentials are given by the equations

$$V_1 = p_{12}e_2, \; V_2 = p_{22}e_2, \ldots V_n = p_{n2}e_2;$$

and it is plain that equations of like form hold good if any one conductor be charged, the others remaining uncharged. In every one of the cases, supposed the potential is constant on the surface of each conductor; if, therefore, we imagine a distribution in which the density at any point is the sum of the densities in all the former cases, we have a distribution consistent with equilibrium, and such that the total charges on the conductors are $e_1, e_2, \ldots e_n$. As the charges are assigned, this (Art 127) is the only possible distribution, and the values $V_1, V_2, \ldots V_n$ of the potential at the conductors, A_1, $A_2, \ldots A_n$ are given by the equations

$$\left.\begin{array}{l} V_1 = p_{11}e_1 + p_{12}e_2 + p_{13}e_3 \ldots p_{1n}e_n \\ V_2 = p_{21}e_1 + p_{22}e_2 + p_{23}e_3 \ldots p_{2n}e_n \\ \vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \\ V_n = p_{n1}e_1 + p_{n2}e_2 \ldots\ldots p_{nn}e_n \end{array}\right\}. \qquad (1)$$

We can now show that $p_{12} = p_{21}$, $p_{23} = p_{32}$, &c., in the following manner :—

If W denote the total energy of the charged system by Art. 50, we have $W = \frac{1}{2}\Sigma e V$, and therefore, by (1) W is a homogeneous quadratic function of the n variables $e_1, \ldots e_n$. Again, if the charge e_1 be increased by an infinitely small amount δe_1, the external work required to bring δe_1 from infinity to the conductor A_1, is $V_1 \delta e_1$, which must therefore denote the increase in the energy of the system due to the increment of the charge e_1. Hence

$$\frac{dW}{de_1}\delta e_1 = V_1 \delta e_1, \quad \text{that is,} \quad \frac{dW}{de_1} = V_1;$$

in like manner,

$$\frac{dW}{de_2} = V_2; \text{ whence } \frac{dV_1}{de_2} = \frac{dV_2}{de_1},$$

and therefore $p_{12} = p_{21}$; similarly $p_{23} = p_{32}$, &c.; accordingly we have

$$\left.\begin{array}{l} V_1 = p_{11}e_1 + p_{12}e_2 + p_{13}e_3 \ldots p_{1n}e_n \\ V_2 = p_{12}e_1 + p_{22}e_2 + p_{23}e_3 \ldots p_{2n}e_n \\ \vdots \\ V_n = p_{1n}e_1 + p_{2n}e_2 \ldots\ldots p_{nn}e_n \end{array}\right\}. \qquad (2)$$

The quantities p_{11}, p_{12}, &c., depend on the forms and relative positions of the conductors, and are called coefficients of potential.

129. **Charges in terms of Potentials.**—By means of (2) the charges e_1, e_2, &c., can be expressed as linear functions of V_1, V_2, &c. Thus W becomes a homogeneous quadratic function of V_1, V_2, &c., and we may write

$$2W_V = q_{11}V_1^2 + q_{22}V_2^2 + 2q_{12}V_1V_2 + \&c. \qquad (3)$$

Again,

$$2W_e = p_{11}e_1^2 + p_{22}e_2^2 + 2p_{12}e_1e_2 + \&c., \qquad (4)$$

and

$$2W = \Sigma e V. \qquad (5)$$

If we suppose each charge to receive a variation, the values V_1, V_2, &c., of the potential receive corresponding variations; and we have from (5)

$$\Sigma V\delta e + \Sigma e\delta V = 2\delta W = \Sigma \frac{dW_e}{de}\,\delta e + \Sigma \frac{dW_v}{dV}\,\delta V,$$

but

$$V_1 = \frac{dW_e}{de_1}, \quad V_2 = \frac{dW_e}{de_2}, \text{ \&c.};$$

whence

$$\Sigma V\delta e = \Sigma \frac{dW_e}{de}\,\delta e,$$

and therefore

$$\Sigma e\delta V = \Sigma \frac{dW_v}{dV}\,\delta V.$$

In this equation the variations δV_1, δV_2, &c., may be regarded as independent and arbitrary, and thus we get

$$e_1 = \frac{dW_v}{dV_1}, \qquad e_2 = \frac{dW_v}{dV_2}, \text{ \&c.} \tag{6}$$

Also, for any variations of the charges and the corresponding variations of the values of the potential, we have

$$\Sigma V\delta e = \Sigma e\delta V. \tag{7}$$

Substituting for the differential coefficients in (6) their values derived from (3) we have

$$\left.\begin{array}{l} e_1 = q_{11}V_1 + q_{12}V_2 \;.\;.\;. + q_{1n}V_n \\ e_2 = q_{12}V_1 + q_{22}V_2 \;.\;.\;. + q_{2n}V_n \\ \vdots \\ e_n = q_{1n}V_1 + q_{2n}V_2 \;.\;.\;. + q_{nn}V_n \end{array}\right\}. \tag{8}$$

The quantities q_{11}, q_{22}, &c. are called coefficients of capacity, and the quantities q_{12}, q_{23}, &c. coefficients of induction.

130. **One Conductor surrounded by another.**—In the investigation of Art. 127 all the conductors have been supposed to be outside one another; but if a conductor A_1 be completely surrounded by another A_2, we may suppose the field $\mathfrak{S}$ to include the space between them, and for the

integral $\int \frac{dV}{d\nu} dS_2$ substitute $\int \left(\frac{dV}{d\nu} dS_2 + \frac{dV}{d\nu'} dS'_2 \right)$,

where S_2 and S'_2 are the outer and inner surfaces of the conductor A_2. The investigation in Art. 127 proceeds then as before.

In this case, if we imagine a closed surface described in the substance of the conductor A_2, at each of its points the resultant force is zero, and therefore so also is the total mass inside this surface. Hence the charge on the inner surface of A_2 is equal in magnitude and opposite in algebraical sign to the charge e_1 on A_1, and is zero if e_1 be zero. Again, if e_1 be zero, the unoccupied region between A_1 and A_2 is bounded by surfaces at each of which the potential is constant, and the integral $\int \frac{dV}{d\nu} dS$ zero, and therefore, by (9), Art. 58, the potential is constant throughout this region, and $V_1 = V_2$.

Since $V_1 = V_2$ we have

$$p_{12} e_2 + p_{13} e_3 \ldots + p_{1n} e_n = p_{22} e_2 + p_{23} e_3 \ldots + p_{2n} e_n$$

for all values of $e_2, e_3, \ldots e_n$, and therefore

$$p_{12} = p_{22}, \quad p_{13} = p_{23}, \ldots p_{1n} = p_{2n}. \tag{9}$$

Also, for any set of charges

$$V_1 - V_2 = (p_{11} - p_{22})\, e_1. \tag{10}$$

Hence, if one conductor be completely surrounded by another, the charge on the inner conductor is proportional to the difference between the values of the potential on it and on the conductor by which it is surrounded.

It follows from (10) that in this case $q_{12} = - q_{11}$, and that $q_{13}, q_{14}, \ldots q_{1n}$ are each zero.

A *condenser* is defined by Clerk Maxwell as a combination of two conductors placed so near together that their coefficient of mutual induction is large. It is, however, usually assumed that the charge on the first conductor is, in this case, proportional to the difference of the potentials on the two conductors. This is *strictly* true when the first conductor is surrounded by the second.

131. **Capacity.**—When a conductor belongs to a specified system, its *capacity* is the ratio of its charge to the value of the potential at its surface, the potential being zero at each of the other conductors.

The capacity of the conductor A_1 belonging to the system $A_1, A_2, \ldots A_n$ is thus the coefficient q_{11} in equations (8).

The *capacity of a conductor not regarded as belonging to a system* is the ratio of its charge to the value of the potential at its surface when there is no other conductor within a finite distance.

In the case of a condenser formed of one conductor A_1 completely surrounded by another A_2, we have

$$e_1 = q_{11}\,(V_1 - V_2),$$

and the value of q_{11} is unaffected by the presence of conductors external to A_2. This appears as follows:—

If e_1 be given, so also is the charge $-e_1$ on the inner surface of A_2. Hence the field $\mathfrak{S}$ between A_1 and A_2 is bounded by surfaces on each of which the potential is constant, and the total charge is given; therefore, as in Art. 127, if V and V' be two possible potentials throughout the field $\mathfrak{S}$ we must have $V = V' +$ constant; whence $V_1 - V_2 = V_1' - V_2'$; and therefore, if e_1 be assigned, so also is $V_1 - V_2$ independently of the state of the field outside A_2. The *capacity of a condenser, such as described above*, is then the ratio of the charge on the inner conductor to the difference of the values of the potential on the two conductors of which the condenser is composed, and this ratio is unaffected by the presence of other conductors.

Examples.

1. Find the capacity of a spherical conductor whose radius is a. *Ans.* a.

2. Find the capacity of a condenser composed of two concentric spheres whose radii are a and b. *Ans.* $\dfrac{ab}{b-a}$. See Ex. 7, Art. 52.

3. Find the capacity per unit of area of a condenser, composed (1°) of two parallel planes, (2°) of two very long coaxal circular cylinders.

Ans. (1°) $\dfrac{1}{4\pi b}$, where b is the distance between the planes;

(2°) $\dfrac{1}{4\pi a}\log\dfrac{b}{a}$, where a and b are the radii of the cylinders.

See Ex. 5 and 10, Art. 52.

4. An insulated uncharged spherical conductor is in the presence of an electrified point; find the potential at the conductor.

In the system composed of the electrified point A_1 and the sphere A_2 the coefficient, p_{11} is infinite, but if $e_1 = 0$ and $e_2 = 0$, we have $V_1 = 0$, and therefore $p_{11}e_1 = 0$ when $e_1 = 0$; hence, if e_1 be zero, $V_1 = p_{12}e_2$; whence $p_{12} = \dfrac{1}{f}$, where f is the distance of A_1 from the centre of the sphere. Again, when e_2 is zero

$$V_2 = p_{12}e_1 = \frac{e_1}{f},$$

which is the potential required, the charge at the electrified point being e_1.

5. An uncharged insulated ellipsoidal conductor is in the presence of an electrified point; find the potential on the conductor.

Applying the method of the last Example, and denoting the electrified point by A_1, when $e_1 = 0$, by Ex. 9, Art. 75, we have

$$p_{12}e_2 = V_1 = e_2\int_{a'}^{\infty}\frac{d\lambda}{\sqrt{(\lambda^2-h^2)(\lambda^2-k^2)}},$$

where a' is the primary semi-axis of the ellipsoid confocal with the conductor which passes through A_1, and h and k are the constants of the confocal system. Hence

$$V_2 = e_1\int_{a'}^{\infty}\frac{d\lambda}{\sqrt{(\lambda^2-h^2)(\lambda^2-k^2)}},$$

where e_1 is the charge at A_1, and V_2, the potential required.

6. An ellipsoidal conductor put to earth is in the vicinity of an electrified point; find the total charge on the conductor.

Here we may employ the equation $\Sigma e'V = \Sigma eV'$, which is a particular case of (22), Art. 51, when the two systems considered in that Article become different states of the same system. In the present case we have

$$e'_1 V_1 + e'_2 V_2 = e_1 V'_1 + e_2 V'_2.$$

In this equation, suppose $e'_1 = 0$, $V_2 = 0$, then e_2 is the charged required, the charge at A_1 being e_1, and we have $e_1 V'_1 + e_2 V'_2 = 0$; but since $e'_1 = 0$, the potential at A_1 is given by the equation

$$V'_1 = e'_2 \int_{a'}^{\infty} \frac{d\lambda}{\sqrt{(\lambda^2 - h^2)(\lambda^2 - k^2)}}, \quad \text{and also} \quad V'_2 = e'_2 \int_{a}^{\infty} \frac{d\lambda}{\sqrt{(\lambda^2 - h^2)(\lambda^2 - k^2)}},$$

where a is the primary semi-axis of the conductor. Substituting, we have

$$e_1 \int_{a'}^{\infty} \frac{d\lambda}{\sqrt{(\lambda^2 - h^2)(\lambda^2 - k^2)}} + e_2 \int_{a}^{\infty} \frac{d\lambda}{\sqrt{(\lambda^2 - h^2)(\lambda^2 - k^2)}} = 0,$$

which determines e_2.

7. Show that, in general, there are n methods of charging a system of n conductors so that the total energy is given, and the value of the potential at each conductor is proportional to the charge.

If we assume $V_1 = \lambda e_1$, $V_2 = \lambda e_2$, &c., we have

$$\begin{aligned}
\lambda e_1 &= p_{11} e_1 + p_{12} e_2 \ldots p_{1n} e_n, \\
\lambda e_2 &= p_{12} e_1 + p_{22} e_2 \ldots p_{2n} e_n, \\
&\vdots \\
\lambda e_n &= p_{1n} e_1 + p_{2n} e_2 \ldots p_{nn} e_n,
\end{aligned}$$

and if λ be a root of the equation

$$\begin{vmatrix}
(p_{11} - \lambda) & p_{12} & \ldots & p_{1n} \\
p_{12} & (p_{22} - \lambda) & \ldots & p_{2n} \\
\vdots & & & \vdots \\
p_{1n} & \ldots & \ldots & (p_{nn} - \lambda)
\end{vmatrix} = 0,$$

we can find corresponding values of $e_1 : e_2$, &c., satisfying the system of n linear equations. Also, by a proper determination of e_1, we can give any assigned value to $\frac{1}{2}(p_{11}e_1^2 + p_{22}e_2^2 + 2p_{12}e_1e_2 + \text{\&c.})$, that is to W the total energy. As the equation in λ has n roots, there are, in general, n systems of values of e_1, e_2, &c.

8. Find the capacity of a conductor formed of the larger segments of two spheres cutting orthogonally.

Ans. $a + b - \dfrac{ab}{\sqrt{a^2 + b^2}}$, where a and b are the radii of the spheres. See (14) Art. 115.

9. Find the capacity of a conductor formed of two equal spheres in contact.

If a denote the radius of one of the spheres, and q the required capacity, by (34), Art. 124, we have $q = 2a \log 2 = 1{\cdot}386294a$.

10. Find the capacity of a conductor formed of a large and a small sphere in contact.

If a denote the radius, and E_a the charge of the small sphere, b and E_b the radius and charge of the large, q the required capacity, L the potential, and E the total charge, by (30) and (32), Art. 124, we find $q = b$, and therefore, if the approximation be not carried beyond $\left(\frac{a}{b}\right)^2$, the capacity is the same as that of the large sphere.

If the approximation be carried on so as to include terms containing $\left(\frac{a}{b}\right)^3$, by (29), Art. 124, we have

$$E_a = Lb\{\nu^2 S_2 - \nu^3(2S_2 - S_3)\},$$

where

$$\nu = \frac{a}{b}, \quad S_2 = \sum_1^\infty \frac{1}{n^2}, \quad S_3 = \sum_1^\infty \frac{1}{n^3}.$$

Again, by (31), Art. 124, we have

$$E_b = Lb\,\{1 - \nu^2 S_2 + \nu^3(2S_2 + S_3)\};$$

whence

$$E = E_a + E_b = Lb(1 + 2\nu^3 S_3).$$

Hence

$$q = b\left(1 + 2S_3\frac{a^3}{b^3}\right).$$

The approximate value of S_3 is $1{\cdot}202$, and therefore

$$q = b\left(1 + 2{\cdot}404\,\frac{a^3}{b^3}\right).$$

132. **Coefficients of Potential.**—The potential energy W of an electrified system is always positive whatever be the charges or the values of the potential, but if all the charges except e_1 be zero $2W = p_{11}e_1^2$, and therefore p_{11} must be positive. Similarly p_{22}, p_{33}, &c., are each positive.

Again, if e_1 be positive and all the charges except e_1 zero, the number of unit tubes of force which terminate on one of the uncharged conductors A_2 must be equal to the number of those which leave it; and as these tubes go from higher to lower potential, the potential at A_2 cannot be the highest or lowest in the field. A similar result holds good for A_3, A_4, . . . A_n, and as the potential cannot be highest or lowest in empty space, the highest potential is on A_1, and the lowest

at infinity, where it is zero. Consequently, the potentials at A_2, A_3, &c., are all positive, but each is less than that at A_1, that is, $p_{12}e_1$, $p_{13}e_1$, . . . $p_{1n}e_1$ are each positive, but less than $p_{11}e_1$. Hence all the coefficients of potential are positive, but any mutual coefficient p_{12} is less than p_{11} or p_{22}.

Besides the conditions specified above p_{11}, p_{12}, p_{22}, &c., must fulfil those belonging to the coefficients of a homogeneous quadratic function of n variables, which is always positive. (See Williamson, *Differential Calculus*, Art. 348.) Some of these are included in those given above.

133. **Coefficients of Capacity and Induction.**—By a method the same as that employed in the preceding Article, it can be shown that the coefficients of capacity q_{11}, q_{22}, &c., are all positive. Also the complete set of coefficients q_{11}, q_{12}, q_{22}, &c., must fulfil the conditions belonging to the coefficients of a positive function.

The coefficients of induction q_{12}, q_{13}, &c., are, however, all negative. Their values are limited by the condition that the sum of those belonging to one conductor is numerically less than its coefficient of capacity, that is,

$$- (q_{12} + q_{13} \ldots + q_{1n}) < q_{11}.$$

To prove this, suppose that the potential is unity on A_1, and zero on every other conductor, then q_{12}, q_{13}, &c. denote the charges on A_2, A_3, &c., or the numbers of unit tubes of force emanating from them.

Since the potential is positive at A_1, and zero at each of the other conductors, it is nowhere less than zero, as it cannot be a minimum in empty space. Accordingly, as the potential is nowhere lower than at A_2, no tubes of force can emanate from A_2, but all those which meet it terminate on it. Hence q_{12} is negative, and so also are q_{13}, q_{14}, &c.

Again, as all the tubes of force emanate from A_1, and may terminate at any of the other conductors or at infinity,

$$- (q_{12} + q_{13} \ldots + q_{1n}) < q_{11}.$$

134. **Conductors surrounded by Another.**—If one or more conductors are surrounded by another, the system has special properties which we proceed to investigate.

Suppose that the conductor A_{m+1} entirely surrounds the region occupied by $A_1 \ldots A_m$; then, if $e_1 = e_2 = \&c. = e_m = 0$, it can be shown, as in Art. 130, that $V_1 = V_2 = \&c. = V_m = V_{m+1}$ for all values of $e_{m+1}, e_{m+2}, \ldots e_n$. Hence

$$\begin{array}{llll} p_{1m+1} = p_{2m+1} = p_{3m+1} = \&c. = p_{m+1\ m+1}, \\ p_{1m+2} = p_{2m+2} = \&c. \ldots\ldots = p_{m+1\ m+2}, \\ \vdots \qquad \vdots \qquad\qquad \vdots \\ p_{1n} = p_{2n} = \&c. \ldots\ldots = p_{m+1\ n}, \end{array}$$

and putting $p_{m+1\ m+1} = P$, we have

$$\left.\begin{array}{l} V_1 - V_{m+1} = (p_{11} - P)\, e_1 + (p_{12} - P)\, e_2 \ldots + (p_{1m} - P) e_m \\ V_2 - V_{m+1} = (p_{12} - P)\, e_1 + (p_{22} - P)\, e_2 \ldots + (p_{2m} - P) e_m \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ V_m - V_{m+1} = (p_{1m} - P)\, e_1 + (p_{2m} - P) e_2 \ldots + (p_{mm} - P) e_m \end{array}\right\} . \quad (11)$$

Equations (11), which determine $V_1 - V_{m+1}$, &c., in terms of $e_1, e_2, \ldots e_m$, are of the same form as those for the absolute potentials of a system of unsurrounded conductors.

By solving (11) we get

$$\left.\begin{array}{l} e_1 = q_{11} (V_1 - V_{m+1}) + q_{12} (V_2 - V_{m+1}) \ldots + q_{1m} (V_m - V_{m+1}) \\ e_2 = q_{12} (V_1 - V_{m+1}) + q_{22} (V_2 - V_{m+1}) \ldots + q_{2m} (V_m - V_{m+1}) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ e_m = q_{1m} (V_1 - V_{m+1}) + q_{2m} (V_2 - V_{m+1}) \ldots + q_{mm} (V_m - V_{m+1}) \end{array}\right\} (12)$$

and therefore, $q_{1m+1} = -\{q_{11} + q_{12} \ldots + q_{1m}\}$, that is,

$$-q_{11} = q_{12} + q_{13} \ldots + q_{1m+1}, \quad -q_{22} = q_{12} + q_{23} \ldots + q_{2m+1}, \ \&c.$$

Also

$$q_{1m+2} = q_{1m+3} = \&c. = q_{1n} = 0, \quad q_{2m+2} = q_{2m+3} = \&c. = q_{2n} = 0, \ \&c. = 0.$$

Again, it appears as in Art. 131, that

$$q_{11},\ q_{12},\ \ldots\ q_{1m},\ q_{22},\ q_{23},\ \ldots\ q_{2m},\ \ldots\ q_{mm}$$

are independent of the state of the field outside A_{m+1}. Hence the charges on the interior conductors are functions of the differences between the values of the potential on them and its value on the surrounding conductor, and are independent of the state of the field external to the latter.

Examples.

1. If there be a system of conductors and a new conductor be brought into the field, the coefficient of potential of any one of the others on itself is diminished.

The final result here is the same as if the portion of space originally unoccupied, and subsequently occupied by the new conductor, were rendered capable of conducting electricity. The consequence of this change would be a new distribution of electricity brought about by the electric forces, and accompanied, therefore, by a diminution of the electric energy of the system. If we now suppose all the conductors uncharged except A_1, the original energy is $\frac{1}{2} p_{11} e_1^2$, and after the introduction of the new conductor the energy becomes $\frac{1}{2} p'_{11} e_1^2$, but, as this is less than the original, we have $p'_{11} < p_{11}$.

2. If two conductors occupying the field be placed in electric communication so as to form a single one, determine the capacity of the new conductor in terms of the coefficients of capacity and induction of the original system.

If we suppose the two conductors originally at the same potential L, this is also the value of the potential on the single conductor formed by their union, and if Q be the capacity of this conductor, we have

$$e_1 = (q_{11} + q_{12})\, L, \quad e_2 = (q_{12} + q_{22})\, L, \quad e_1 + e_2 = QL\,;$$

whence

$$Q = q_{11} + q_{22} + 2q_{12}.$$

3. Show that the capacity of any conductor is less than that of another conductor geometrically capable of surrounding the former.

If A_2, supposed non-conducting, were made to surround A_1, no change would take place in the electric condition of A_1. If then electric communication between A_1 and A_2 were established, and A_2 rendered conducting, the electric charge on A_1 would be transferred to the external surface of A_2, and in effecting this transference the electric forces would do work, and therefore the electric energy would be diminished. Hence if e be the charge originally on A_1, we have

$$\frac{e^2}{q_2} < \frac{e^2}{q_1},$$

and therefore, $q_2 > q_1$, where q_1 and q_2 are the capacities of A_1 and A_2.

135. **Effect of Displacements on Energy.**—If ξ be a generalized coordinate on which the relative positions of the conductors, A_1, A_2, &c., depend, the charges e_1, e_2, &c., are independent of ξ, but if ξ vary, so do the values of the potential at the conductors as well as the coefficients p_{11}, p_{12}, &c., q_{11}, q_{12}, &c.; then

$$\frac{dW_e}{d\xi} = \tfrac{1}{2}\left(\frac{dp_{11}}{d\xi}e_1^2 + 2\frac{dp_{12}}{d\xi}e_1e_2 + \frac{dp_{22}}{d\xi}e_2^2 + \&c.\right),$$

$$\frac{d}{d\xi}W_V = \tfrac{1}{2}\left(\frac{dq_{11}}{d\xi}V_1^2 + 2\frac{dq_{12}}{d\xi}V_1V_2 + \frac{dq_{22}}{d\xi}V_2^2 + \&c.\right) + \Sigma\frac{dW_V}{dV}\frac{dV}{d\xi}.$$

The first part of the expression for $\frac{d}{d\xi}W_V$ in which the values V_1, V_2, &c., of the potential are not supposed to vary, may be denoted by $\frac{dW_V}{d\xi}$; and substituting for $\frac{dW_V}{dV_1}$, $\frac{dW_V}{dV_2}$, &c., their values given by (6), Art. 129, we have

$$\frac{d}{d\xi}W_V = \frac{dW_V}{d\xi} + \Sigma e\frac{dV}{d\xi}.$$

When the charges are invariable from (5), Art. 129, we have $2\delta W = \Sigma e\delta V$, and therefore if the values of the potential and of W vary in consequence of a variation in ξ, we get

$$\delta W = \frac{dW_V}{d\xi}\delta\xi + \Sigma e\delta V = \frac{dW_V}{d\xi}\delta\xi + 2\delta W;$$

whence

$$\frac{dW_V}{d\xi}\delta\xi = -\delta W = -\frac{dW_e}{d\xi}\delta\xi. \qquad (13)$$

Equations (6) and (13) can be obtained directly from the three expressions for W by supposing e_1, e_2, . . . e_n, V_1, V_2, . . . V_n, and ξ all to vary; then,

$$\Sigma V\delta e + \Sigma e\delta V = 2\delta W = \delta W_e + \delta W_V$$
$$= \Sigma\frac{dW}{de}\delta e + \Sigma\frac{dW_V}{dV}\delta V + \left(\frac{dW_e}{d\xi} + \frac{dW_V}{d\xi}\right)\delta\xi.$$

Now, by Art. 128, we have

$$V_1 = \frac{dW_e}{de_1}, \quad V_2 = \frac{dW_e}{de_2}, \text{ \&c.,}$$

and therefore the terms containing δe_1, δe_2, &c., vanish, and the variations of the $(n + 1)$ variables $V_1, V_2, \ldots V_n$, and ξ, may be regarded as independent and arbitrary; whence we have

$$e_1 = \frac{dW_V}{dV_1}, \quad e_2 = \frac{dW_V}{dV_2}, \text{ \&c.,}$$

and

$$\frac{dW_e}{d\xi} + \frac{dW_V}{d\xi} = 0. \tag{14}$$

136. **Forces between Conductors.**—If Ξ be the generalized component of force due to electric action which tends to alter the coordinate ξ, if this coordinate receive an increment $\delta\xi$ the work done by Ξ is $\Xi\delta\xi$, and this must be equal to the diminution of potential energy; whence

$$\Xi = -\frac{dW_e}{d\xi} = \frac{dW_V}{d\xi}. \tag{15}$$

If in the position of the system in which ξ has become $\xi + \delta\xi$, the values of the potential were the same as those in the original position, the potential energy of the system would be $$W + \frac{dW_V}{d\xi}\delta\xi.$$

If the electric condition of the system be unaltered by any external cause, the charges remain constant; and after the displacement $\delta\xi$, the energy becomes $W + \frac{dW_e}{d\xi}d\xi$, that is, by (13), it becomes $W - \frac{dW_V}{d\xi}\delta\xi$. Hence, if during the displacement $\delta\xi$ the values of the potential at the conductors be maintained constant by an external source, the energy supplied is $2\frac{dW_V}{d\xi}\delta\xi$, which, by (15), is twice the work done by the electric forces of the system in the displacement.

Examples.

1. Find the electric energy due to two charged spheres at an infinite distance apart.

In this case if $e_2 = 0$, we have $V_2 = 0$, and therefore $p_{12} = 0$, also

$$p_{11} = \frac{1}{a}, \quad p_{22} = \frac{1}{b},$$

where a and b are the radii of the spheres. Hence

$$W = \tfrac{1}{2}\left(\frac{e_1^2}{a} + \frac{e_2^2}{b}\right).$$

2. Find the work M required to bring together, from an infinite distance, two equal charged spheres.

If a denote the radius of one of the spheres, the potential electric energy W when the spheres are at an infinite distance apart is given by the equation $2W = \dfrac{e_1^2 + e_2^2}{a}$, and when they are in contact by the equation $2W' = \dfrac{(e_1 + e_2)^2}{2a \log 2}$. See Ex. 9, Art. 131. Since $M = W' - W$, we have, therefore,

$$M = \frac{1}{4a \log 2}\left\{2e_1e_2 - (2 \log 2 - 1)(e_1^2 + e_2^2)\right\};$$

and substituting for log 2 its approximate value 0·693, we get

$$M = \frac{e_1^2}{a}\left\{\frac{1}{1·386}\,\frac{e_2}{e_1} - 0·14\left(1 + \frac{e_2^2}{e_1^2}\right)\right\}.$$

M is approximately zero if e_1 and e_2 have like signs and $\dfrac{e_2}{e_1} = 5$. If $e_2 : e_1 > 5$, the value of M is negative, and the spheres tend to approach each other without the expenditure of any external work.

If $e_2 : e_1 < 5$, the value of M is positive till $e_2 : e_1 = 1 : 5$, when M is again zero; and if $e_2 : e_1 < 1 : 5$, the value of M is negative. The last two results are of course an immediate consequence of the former.

When e_1 and e_2 have unlike signs, M is always negative.

3. Find the energy due to a charged conductor formed of a large and a small sphere in contact.

If a denote the radius of the small sphere, b that of the large, E the total charge, and W the energy, we have, Ex. 10, Art. 131,

$$W = \frac{E^2}{2b}\left(1 + 2S_3\left(\frac{a}{b}\right)^3\right)^{-1} = \frac{E^2}{b}\left(\tfrac{1}{2} - 1·202\left(\frac{a}{b}\right)^3\right).$$

4. Find the work required to bring together, from an infinite distance, a small and a large sphere, each charged with a given quantity of electricity.

If e_1 and e_2 denote the charges on the small and the large sphere respectively, and M the work required, we have

$$M = \frac{(e_1 + e_2)^2}{b} \left\{\tfrac{1}{2} - 1{\cdot}202 \left(\frac{a}{b}\right)^3\right\} - \frac{e_1^2}{2a} - \frac{e_2^2}{2b}$$

$$= \frac{e_1 e_2}{b} \left\{1 - 2{\cdot}404 \left(\frac{a}{b}\right)^3\right\} - \frac{e_1^2}{2b} \left\{\frac{b}{a} - 1 + 2{\cdot}404 \left(\frac{a}{b}\right)^3\right\} - 1{\cdot}202 \frac{e_2^2}{b} \left(\frac{a}{b}\right)^3.$$

If the small sphere be originally uncharged, $e_1 = 0$, and

$$M = -1{\cdot}202 \frac{e_2^2}{b} \left(\frac{a}{b}\right)^3.$$

Hence the electrical forces of the system will do work in bringing a small uncharged sphere from an infinite distance into contact with a large one which is charged.

5. Find the mutual repulsion between two charged insulated spheres on which the values of the potential are V_1 and V_2.

If c denote the distance between the centres of the spheres, W the energy of the charged system, and F the required force, by (15),

$$F = \frac{dW_V}{dc} = \tfrac{1}{2} \frac{dq_{11}}{dc} V_1^2 + \frac{dq_{12}}{dc} V_1 V_2 + \tfrac{1}{2} \frac{dq_{22}}{dc} V_2^2.$$

The values of q_{11}, q_{12}, and q_{22} are given by (52), Art. 126; and by Ex. 8, Art. 126, we have

$$\lambda = \frac{\sqrt{(a^2 + k^2)} - k}{a} = \frac{c^2 + a^2 - b^2 - 2ck}{2ac},$$

$$\mu = \frac{\sqrt{(b^2 + k^2)} - k}{b} = \frac{c^2 + b^2 - a^2 - 2ck}{2bc},$$

$$\nu = \frac{c^2 - a^2 - b^2 - 2ck}{2ab} = \lambda\mu, \quad c = \sqrt{(a^2 + k^2)} + \sqrt{(b^2 + k^2)}.$$

Hence $a \dfrac{d\lambda}{dk} = \dfrac{k}{\sqrt{(a^2 + k^2)}} - 1$, and therefore $\dfrac{d\lambda}{dk} = \dfrac{-\lambda}{\sqrt{(a^2 + k^2)}}.$

In like manner $\dfrac{d\mu}{dk} = \dfrac{-\mu}{\sqrt{(b^2 + k^2)}}.$ Again

$$\frac{d\nu}{dk} = \mu \frac{d\lambda}{dk} + \lambda \frac{d\mu}{dk} = -\nu \left\{\frac{1}{\sqrt{(a^2 + k^2)}} + \frac{1}{\sqrt{(b^2 + k^2)}}\right\} = \frac{-\nu c}{\sqrt{(a^2 + k^2)(b^2 + k^2)}},$$

$$\frac{d(\lambda\nu)}{dk} = \lambda \frac{d\nu}{dk} + \nu \frac{d\lambda}{dk} = \frac{-\lambda\nu \{c + \sqrt{(b^2 + k^2)}\}}{\sqrt{(a^2 + k^2)(b^2 + k^2)}},$$

$$\frac{dc}{dk} = \frac{k}{\sqrt{(a^2 + k^2)}} + \frac{k}{\sqrt{(b^2 + k^2)}}; \quad \text{whence} \quad \frac{dk}{dc} = \frac{\sqrt{(a^2 + k^2)(b^2 + k^2)}}{ck},$$

and therefore we obtain

$$\frac{d(\lambda\nu)}{dc} = -\frac{\lambda\nu}{k}\left(1 + \frac{\sqrt{(b^2+k^2)}}{c}\right), \quad \frac{d\nu}{dc} = \frac{-\nu}{k}.$$

We have, then,

$$\frac{1}{2}\frac{dq_{11}}{dc} = \frac{\sqrt{(a^2+k^2)(b^2+k^2)}}{ck}\sum_0^\infty \frac{(\lambda\nu)^{2n+1}}{1-\nu^{2n+1}}$$

$$-\left(1 + \frac{\sqrt{(b^2+k^2)}}{c}\right)\sum_0^\infty (2n+1)\frac{(\lambda\nu)^{2n+1}}{1-\nu^{2n+1}}$$

$$-\nu\sum_0^\infty (2n+1)\,\nu^{2n}\frac{(\lambda\nu)^{2n+1}}{(1-\nu^{2n+1})^2},$$

with a similar expression for $\frac{1}{2}\frac{dq_{22}}{dc}$; and, again, we have

$$-\frac{dq_{12}}{dc} = 2\frac{\sqrt{(a^2+k^2)(b^2+k^2)}}{ck}\sum_0^\infty \frac{\nu^{2n+1}}{1-\nu^{2n+1}}$$

$$-2\nu\sum_0^\infty (2n+1)\,\nu^{2n}\left(\frac{1}{1-\nu^{2n+1}} + \frac{\nu^{2n+1}}{(1-\nu^{2n+1})^2}\right).$$

Combining the first term with the third, and the second with the last, in the expressions for $\frac{1}{2}\frac{dq_{11}}{dc}$ and $\frac{1}{2}\frac{dq_{22}}{dc}$, and combining the last two terms in that for $\frac{dq_{12}}{dc}$, we get

$$\frac{1}{2}\frac{dq_{11}}{dc} = \frac{\sqrt{(a^2+k^2)(b^2+k^2)}}{ck}\sum_0^\infty\left(1 - \frac{(2n+1)k}{\sqrt{(a^2+k^2)}}\right)\frac{(\lambda\nu)^{2n+1}}{1-\nu^{2n+1}} - \sum_0^\infty (2n+1)\frac{(\lambda\nu)^{2n+1}}{(1-\nu^{2n+1})^2},$$

$$\frac{dq_{12}}{dc} = -2\frac{\sqrt{(a^2+k^2)(b^2+k^2)}}{ck}\sum_0^\infty \frac{\nu^{2n+1}}{1-\nu^{2n+1}} + 2\sum_0^\infty (2n+1)\frac{\nu^{2n+1}}{(1-\nu^{2n+1})^2},$$

$$\frac{1}{2}\frac{dq_{22}}{dc} = \frac{\sqrt{(a+k^2)(b^2+k^2)}}{ck}\sum_0^\infty\left(1 - \frac{(2n+1)k}{\sqrt{(b^2+k^2)}}\right)\frac{(\mu\nu)^{2n+1}}{1-\nu^{2n+1}} - \sum_0^\infty (2n+1)\frac{(\mu\nu)^{2n+1}}{(1-\nu^{2n+1})^2}.$$

These expressions are due to Mr. F. Purser.

The differential coefficients of q_{11}, &c., may otherwise be obtained from the values of E_a and E_b given in Ex. 6, Art. 126. From these we get

$$q_{11} = k\sum_0^\infty \frac{1}{\sinh(\alpha+n\gamma)}, \quad q_{12} = -k\sum_1^\infty \frac{1}{\sinh n\gamma}, \quad q_{22} = k\sum_0^\infty \frac{1}{\sinh(\beta+n\gamma)},$$

where

$$\sinh\alpha = \frac{k}{a}, \qquad \sinh\beta = \frac{k}{b}, \qquad \gamma = \alpha + \beta,$$

$$a\cosh\alpha = \sqrt{(a^2 + k^2)} = \frac{c^2 + a^2 - b^2}{2c} = \frac{c}{2} + \frac{a^2 - b^2}{2c},$$

$$b\cosh\beta = \sqrt{(b^2 + k^2)} = \frac{c^2 + b^2 - a^2}{2c} = \frac{c}{2} + \frac{b^2 - a^2}{2c}.$$

We have then

$$\frac{dk}{dc} = \frac{ab\cosh\alpha\cosh\beta}{ck}, \quad \frac{d\alpha}{dc} = \frac{b\cosh\beta}{ck}, \quad \frac{d\beta}{dc} = \frac{a\cosh\alpha}{ck}, \quad \frac{d\gamma}{dc} = \frac{1}{k};$$

whence

$$\frac{1}{2}\frac{dq_{11}}{dc} = \frac{ab\cosh\alpha\cosh\beta}{2ck}\sum_0^\infty \frac{1}{\sinh(\alpha+n\gamma)} - \sum_0^\infty \frac{b\cosh\beta + nc}{2c}\,\frac{\cosh(\alpha+n\gamma)}{\sinh^2(\alpha+n\gamma)},$$

$$\frac{dq_{12}}{dc} = -\frac{ab\cosh\alpha\cosh\beta}{ck}\sum_1^\infty \frac{1}{\sinh n\gamma} + \sum_1^\infty \frac{n\cosh n\gamma}{\sinh^2 n\gamma},$$

$$\frac{1}{2}\frac{dq_{22}}{dc} = \frac{ab\cosh\alpha\cosh\beta}{2ck}\sum_0^\infty \frac{1}{\sinh(\beta+n\gamma)} - \sum_0^\infty \frac{a\cosh\alpha + nc}{2c}\,\frac{\cosh(\beta+n\gamma)}{\sinh^2(\beta+n\gamma)}.$$

6. From the values of q_{11}, q_{12}, and q_{22}, given in Ex. 1, Art. 126, find the leading terms in the expression for the mutual force F between two charged spheres.

$$F = \tfrac{1}{2}\frac{dq_{11}}{dc}V_1^2 + \frac{dq_{12}}{dc}V_1V_2 + \tfrac{1}{2}\frac{dq_{22}}{dc}V_2^2$$

$$= -V_1^2\left\{\frac{a^2bc}{(c^2-b^2)^2} + \frac{a^3b^2c(2c^2-2b^2-a^2)}{(c^2-b^2+ac)^2(c^2-b^2-ac)^2}\right\}$$

$$+ V_1V_2\left\{\frac{ab}{c^2} + \frac{a^2b^2(3c^2-a^2-b^2)}{c^2(c^2-a^2-b^2)^2} + \frac{a^3b^3\{(c^2-a^2-b^2)(5c^2-a^2-b^2)-a^2b^2\}}{c^2(c^2-a^2-b^2+ab)^2(c^2-a^2-b^2-ab)^2}\right\}$$

$$- V_2^2\left\{\frac{ab^2c}{(c^2-a^2)^2} + \frac{a^2b^3c(2c^2-2a^2-b^2)}{(c^2-a^2+bc)^2(c^2-a^2-bc)^2}\right\}$$

7. Show how to exhibit the series expressing the coefficients of capacity and induction of two electrified spheres as rational functions of their radii and the distance between their centres.

By (48), Art. 126, we have

$$q_{11} = a + a(1 - \lambda^2) \sum_1^\infty \frac{\nu^n}{1 - \lambda^2 \nu^{2n}};$$

but it is shown, in Art. 126, that

$$\lambda^2 = \frac{\nu a + \nu^2 b}{\nu a + b}, \quad \text{and that} \quad 1 - \lambda^2 = \frac{2\nu ck}{a(\nu a + b)};$$

whence, by substitution, we get

$$q_{11} = a + \frac{2\nu ck}{\nu a + b} \sum \frac{\nu^n (\nu a + b)}{\nu a(1 - \nu^{2n}) + b(1 - \nu^{2n+2})}$$

$$= a + \frac{2\nu ck}{1 - \nu^2} \sum_1^\infty \frac{\nu^n}{\nu a(1 + \nu^2 + \ldots + \nu^{2(n-1)}) + b(1 + \nu^2 \ldots + \nu^{2n})}.$$

By Art. 126, we have $\dfrac{2\nu ck}{1 - \nu^2} = ab$, and, therefore, we get

$$q_{11} = a + ab \sum_1^\infty \frac{1}{a(\nu^{n-1} + \nu^{-(n-1)} + \nu^{n-3} + \nu^{-(n-3)} + \&c.) + b(\nu^n + \nu^{-n} + \&c.)}.$$

Again

$$q_{12} = -\frac{ab}{c} \sum_0^\infty \frac{\nu^n(1 - \nu^2)}{1 - \nu^{2n+2}}$$

$$= -\frac{ab}{c} \sum \frac{\nu^n}{1 + \nu^2 \ldots + \nu^{2n}} = -\frac{ab}{c} \sum_0^\infty \frac{1}{(\nu^n + \nu^{-n} + \&c.)}.$$

Hence putting $\nu^n + \nu^{-n} = S_n$, we have

$$q_{11} = a + ab \sum_1^\infty \frac{1}{a(S_{n-1} + S_{n-3} + \&c.) + b(S_n + S_{n-2} + \&c.)},$$

$$q_{12} = -\frac{ab}{c} \sum_0^\infty \frac{1}{S_n + S_{n-2} + \&c.}.$$

It is to be observed that in these expressions for q_{11} and q_{12} we must put unity for S_0, and that the series then terminates.

q_{22} is obtained from q_{11} by interchanging a and b.

8. Show how to express the force between two charged spheres as a rational function of their radii, and the distance between their centres.

If we differentiate the expressions for q_{11}, &c., given in the last Example, we obtain a series of rational quantities multiplied each by a term of the form

$$\frac{d}{dc}(\nu^n + \nu^{-n}),$$

but

$$\frac{d}{dc}(\nu^n + \nu^{-n}) = n(\nu^{n-1} - \nu^{-(n+1)}) \frac{d\nu}{dc};$$

and, as $\dfrac{d\nu}{dc} = -\dfrac{\nu}{k}$ by Ex. 5, we get

$$\frac{d}{dc}(\nu^n + \nu^{-n}) = -\frac{n}{k}(\nu^n - \nu^{-n}) = \frac{n\nu^{-n}}{k}(1 - \nu^{2n})$$
$$= \frac{n(1-\nu^2)}{\nu k}(\nu^{n-1} + \nu^{-(n-1)} + \&c.) = \frac{2nc}{ab}(\nu^{n-1} + \nu^{-(n-1)} + \&c.),$$

since by Art. 126, we have $\dfrac{1-\nu^2}{\nu k} = \dfrac{2c}{ab}$.

Hence, we get $\dfrac{dS_n}{dc} = \dfrac{2nc}{ab}(S_{n-1} + S_{n-3} + \&c.)$, and therefore,

$$\tfrac{1}{2}\frac{dq_{11}}{dc}$$
$$= -c\sum_1^\infty \frac{a\sum_0^m (n-m-1)(m+1)\,S_{n-2(m+1)} + b\sum_0^m (n-m)(m+1)\,S_{n-(2m+1)}}{\{a(S_{n-1} + S_{n-3} + \&c.) + b(S_n + S_{n-2} + \&c.)\}^2}$$

$$\frac{dq_{12}}{dc} = \frac{ab}{c^2}\sum_0 \frac{1}{S_n + S_{n-2} + \&c.} + 2\sum_1^\infty \frac{\sum_0^m (n-m)(m+1)\,S_{n-(2m+1)}}{(S_n + S_{n-2} + \&c.)^2},$$

where unity is to be substituted for S_0, and each finite series terminates when the suffix of S is 1 or 0.

9. Show that, in the case of two equal spheres in contact, $\dfrac{dq_{11}}{dc} + \dfrac{dq_{12}}{dc}$ is reducible to the value $\frac{1}{8}(\log 2 - \frac{1}{4})$.

In this case $b = a$, $c = 2a$, $\nu = 1$, $S_n = 2$; and, by Ex. 8, we have

$$\frac{dq_{11}}{dc} + \frac{dq_{12}}{dc}$$
$$= -4\sum_1^\infty \frac{n(S_{n-1} + S_{n-3} + \&c.) + (n-1)(S_{n-2} + S_{n-4} + \&c.) + \&c.}{(S_n + S_{n-1} + \ldots + S_1 + 1)^2}$$
$$+ \tfrac{1}{4}\sum_0^\infty \frac{1}{S_n + S_{n-2} + \&c.}$$
$$+ 2\sum_1^\infty \frac{n(S_{n-1} + S_{n-3} + \&c.) + (n-2)(S_{n-3} + S_{n-5} + \&c.) + \&c.}{(S_n + S_{n-2} + \&c.)^2}.$$

If n be odd, the series $S_{n-1} + S_{n-3} + \&c.$ terminates with $S_2 + 1$, and is equal to $2\dfrac{n-1}{2} + 1$, that is to n. If n be even, the series terminates with S_1, and is equal to $2\dfrac{n}{2}$, which is also n. Hence, whether n be odd or even, $n(S_{n-1} + S_{n-3} + \&c.) \equiv n^2$. Again $S_n + S_{n-1} \ldots + S_1 + 1 = 2n + 1$, and $S_n + S_{n-2} + \&c. = 2\dfrac{n}{2} + 1$, or $2\dfrac{n+1}{2}$, according as n is even or odd, and in either case the sum of the series is $n + 1$. Hence we obtain

$$\frac{dq_{11}}{dc} + \frac{dq_{12}}{dc} = -4\sum_1^\infty \frac{\sum_1^n m^2}{(2n+1)^2} + \tfrac{1}{4}\sum_0^\infty \frac{1}{n+1} + 2\sum_1^\infty \frac{n^2 + (n-2)^2 + \&c.}{(n+1)^2}.$$

As is well known

$$\sum_1^n m^2 = \frac{n(n+1)(2n+1)}{6}, \text{ and } \quad n^2 + (n-2)^2 + \&c. = \frac{n(n+1)(n+2)}{6}$$

whether n be even or odd. Again

$$4\sum_1^\infty \frac{n(n+1)(2n+1)}{6(2n+1)^2} = \sum_1^\infty \frac{4n^2+4n}{6(2n+1)} = \sum_1^\infty \frac{(2n+1)^2-1}{6(2n+1)},$$

$$2\sum_1^\infty \frac{n(n+1)(n+2)}{6(n+1)^2} = \sum_1^\infty \frac{2(n^2+2n)}{6(n+1)} = \sum_1^\infty \frac{2(n+1)^2-2}{6(n+1)}.$$

Accordingly we have

$$\frac{dq_{11}}{dc} + \frac{dq_{12}}{dc}$$
$$= \tfrac{1}{6}\left\{\sum_1^\infty (2n+2) - \sum_1^\infty (2n+1)\right\} + \tfrac{1}{6}\sum_1^\infty \frac{1}{2n+1} + \tfrac{1}{4} + (\tfrac{1}{2} - \tfrac{2}{3})\sum_1^\infty \frac{1}{2n+2}$$
$$= \tfrac{1}{6}\log(2 - \tfrac{1}{2}) + \tfrac{1}{4} - \frac{\xi}{6}, \text{ where } \xi = \sum_1^\infty (2n+1) - \sum_1^\infty (2n+2).$$

In order to find the value of ξ we may proceed as follows :—

If θ be less than 1, we have $\dfrac{\theta}{1-\theta^2} = \theta + \theta^3 + \theta^5 + \&c.$;

whence $$1 + 3\theta^2 + 5\theta^4 + \&c. = \frac{d}{d\theta}\frac{\theta}{1-\theta^2} = \frac{1+\theta^2}{(1-\theta^2)^2}.$$

Also $$2\theta + 4\theta^3 + \&c. = \frac{d}{d\theta}\frac{1}{1-\theta^2} = \frac{2\theta}{(1-\theta^2)^2}.$$

Hence, $$1 + 3\theta^2 + 5\theta^4 + \&c. - (2\theta + 4\theta^3 + \&c.) = \frac{(1-\theta)^2}{(1-\theta^2)^2} = \frac{1}{(1+\theta)^2},$$

and making θ equal to 1, we obtain $\xi + 1 - 2 = \frac{1}{4}$; whence $\xi = \frac{5}{4}$, and we get

$$\frac{dq_{11}}{dc} + \frac{dq_{12}}{dc} = \tfrac{1}{6}\log 2 - \tfrac{1}{12} + \tfrac{1}{4} - \tfrac{5}{24} = \tfrac{1}{6}(\log 2 - \tfrac{1}{4}).$$

10. Show that $\frac{1}{6}(\log 2 - \frac{1}{4})L^2$ denotes the repulsive force between two equal spheres in contact and charged to potential L.

11. If two charged spherical conductors (A) and (B) influence each other, if Σ_1 and Σ_2 denote the systems of charges inside (A) and (B), respectively, which would produce the actual potential in external space, and if u be the potential due to Σ_1, and v that due to Σ_2, show that u is equal to the potential in external space due to the distribution on (A), and v equal to that due to the distribution on (B).

A distribution on the surface of (A), whose potential is u at each point of this surface, produces a potential u throughout the whole of space. Hence, by Art. 127, distributions on (A) and (B), producing potentials u and v, must be the actual distributions.

12. Two equal spherical conductors are charged to the same potential L; deduce an expression for the force between them from the values for q_{11} and q_{12} given by (48), Art. 126.

In this case

$$a = b, \quad \frac{c^2}{a^2} = \nu + \frac{1}{\nu} + 2 = \frac{(1+\nu)^2}{\nu}, \quad \lambda^2 = \frac{a^2(1+\nu)^2}{c^2} = \nu;$$

whence $\quad c = \dfrac{a(1+\lambda^2)}{\lambda} = a\left(\lambda + \dfrac{1}{\lambda}\right)$; and from (48), Art. 126, we have

$$q_{11} = a(1-\lambda^2)\sum_0^\infty \frac{\lambda^{2n}}{1-\lambda^{4n+2}} = \frac{a(1-\lambda^2)}{\lambda}\sum_0^\infty \frac{\lambda^{2n+1}}{1-\lambda^{4n+2}},$$

$$-q_{12} = \frac{a(1-\lambda^4)\lambda}{1+\lambda^2}\sum_0^\infty \frac{\lambda^{2n}}{1-\lambda^{4n+4}} = \frac{a(1-\lambda^2)}{\lambda}\sum_0^\infty \frac{\lambda^{2n+2}}{1-\lambda^{4n+4}};$$

and therefore,

$$q_{11} + q_{12} = \frac{a(1-\lambda^2)}{\lambda}\sum_1^\infty (-1)^{n+1}\frac{\lambda^n}{1-\lambda^{2n}}.$$

If F denote the force between the conductors, we have

$$F = L^2\frac{d(q_{11}+q_{12})}{dc} = -\frac{\lambda^2}{a(1-\lambda^2)}L^2\frac{d(q_{11}+q_{12})}{d\lambda},$$

and, putting

$$Q = \sum_1^\infty (-1)^{n+1}\frac{\lambda^n(1-\lambda^2)}{1-\lambda^{2n}},$$

we get

$$F = \frac{-\lambda^2}{1-\lambda^2}L^2\frac{d}{d\lambda}\left(\frac{Q}{\lambda}\right) = \frac{L^2}{1-\lambda^2}\left(Q - \lambda\frac{dQ}{d\lambda}\right).$$

13. Find the value of the force between two equal spherical conductors at potential L, when they are in contact.

If we put $\lambda^2 = 1 - \xi^2$, we have

$$\lambda^{2n} = 1 - n\xi^2 + \frac{n(n-1)}{2}\xi^4 - \frac{n(n-1)(n-2)}{6}\xi^6 + \&c.,$$

and

$$(1-\lambda^{2n})^{-1} = \frac{1}{n\xi^2}\left\{1 + \frac{n-1}{2}\xi^2 + \left[\frac{(n-1)^2}{4} - \frac{(n-1)(n-2)}{6}\right]\xi^4 + \&c.\right\}$$

$$= \frac{1}{\xi^2}\left\{\frac{1}{n} + \frac{n-1}{2n}\xi^2 + \frac{n^2-1}{12n}\xi^4 + \&c.\right\}.$$

Hence, we get

$$Q = \sum_1^\infty (-1)^{n+1}\lambda^n\left\{\frac{1}{n} + \frac{1}{2}\left(1 - \frac{1}{n}\right)(1-\lambda^2) + \frac{1}{12}\left(n - \frac{1}{n}\right)(1-\lambda^2)^2 + \&c.\right\};$$

but $\quad \Sigma(-1)^{n+1}\dfrac{\lambda^n}{n} = \log(1+\lambda), \qquad \Sigma(-1)^{n+1}\lambda^n = \dfrac{\lambda}{1+\lambda},$

$$\Sigma(-1)^{n+1}n\lambda^n = \lambda\frac{d}{d\lambda}\frac{\lambda}{1+\lambda} = \frac{\lambda}{(1+\lambda)^2};$$

and substituting, we have

$$Q = \left(1 - \frac{1-\lambda^2}{2} - \frac{(1-\lambda^2)^2}{12}\right)\log(1+\lambda) + \frac{1}{2}(1-\lambda^2)\,\frac{\lambda}{1+\lambda}$$
$$+\frac{1}{12}(1-\lambda^2)^2\,\frac{\lambda}{(1+\lambda)^2} + \&c.$$
$$= \frac{5+8\lambda^2-\lambda^4}{12}\log(1+\lambda) + \frac{7\lambda}{12} - \frac{2\lambda^2}{3} + \frac{\lambda^3}{12} + \&c.$$

Differentiating, we obtain

$$\lambda\frac{dQ}{d\lambda} = \frac{16\lambda^2-4\lambda^4}{12}\log(1+\lambda) + \frac{5\lambda+8\lambda^3-\lambda^5}{12(1+\lambda)} + \frac{7\lambda}{12} - \frac{4\lambda^2}{3} + \frac{3\lambda^3}{12} + \&c.$$

Hence we have

$$Q - \lambda\frac{dQ}{d\lambda} = \frac{5(1-\lambda^2) - 3\lambda^2(1-\lambda^2)}{12}\log(1+\lambda)$$
$$- \frac{5\lambda + 8\lambda^3 - \lambda^5 - 8\lambda^2(1+\lambda) + 2\lambda^3(1+\lambda)}{12(1+\lambda)} + \&c.$$

It is easy to see that the numerator of the fraction in the second term of this expression may be put into the form

$$\lambda\left\{1-\lambda^4+4(1-\lambda)-2\lambda(1-\lambda)(2+\lambda)\right\};$$

and therefore we get

$$\frac{1}{1-\lambda^2}\left(Q-\lambda\frac{dQ}{d\lambda}\right) = \frac{5-3\lambda^2}{12}\log(1+\lambda)$$
$$-\frac{\lambda}{12(1+\lambda)^2}\left\{(1+\lambda)(1+\lambda^2)+4-2\lambda(2+\lambda)\right\} + \&c.$$

When $\lambda = 1$, and $\xi = 0$, this expression becomes

$$\frac{5-3}{12}\log 2 - \frac{1}{48}(4+4-6), \quad \text{that is,} \quad \frac{1}{6}\left(\log 2 - \frac{1}{4}\right).$$

Hence

$$F = \frac{L^2}{6}\left(\log 2 - \frac{1}{}\right).$$

The mode of investigation adopted in this Example is due to Mr. F. Purser.

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[*The figures refer to the pages.*]

THE END.

theory of attraction.